Entropy and Information Theory

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Second Edition

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to Tim, Lori, Julia, Peter,
Gus, Amy, and Alice

and in memory of Tino

Preface

This book is devoted to the theory of probabilistic information measures and their application to coding theorems for information sources and noisy channels, with a strong emphasis on source coding and stationary codes. The eventual goal is a general development of Shannon's mathematical theory of communication for single user systems, but much of the space is devoted to the tools and methods required to prove the Shannon coding theorems, especially the notions of sources, channels, codes, entropy, information, and the entropy ergodic theorem. These tools form an area common to ergodic theory and information theory and comprise several quantitative notions of the information in random variables, random processes, and dynamical systems. Examples are entropy, mutual information, conditional entropy, conditional information, and relative entropy (discrimination, Kullback-Leibler information, informational divergence), along with the limiting normalized versions of these quantities such as entropy rate and information rate. In addition to information we will be concerned with the distance or distortion between the random objects, that is, the accuracy of the representation of one random object by another or the degree of mutual approximation. Much of the book is concerned with the properties of these quantities, especially the long term asymptotic behavior of average information and distortion, where both sample averages and probabilistic averages are of interest.

The book has been strongly influenced by M. S. Pinsker's classic *Information and Information Stability of Random Variables and Processes* and by the seminal work of A. N. Kolmogorov, I. M. Gelfand, A. M. Yaglom, and R. L. Dobrushin on information measures for abstract alphabets and their convergence properties. The book also has as a major influence the work of D.S. Ornstein on the isomorphism problem in ergodic theory, especially on his ideas of stationary codes mimicking block codes implied by the entropy ergodic theorem and of the d-bar distance between random processes. Many of the results herein are extensions of their

generalizations of Shannon's original results. The mathematical models adopted here are more general than traditional treatments in that nonstationary and nonergodic information processes are treated. The models are somewhat less general than those of the Russian school of information theory in the sense that standard alphabets rather than completely abstract alphabets are considered. This restriction, however, permits many stronger results as well as the extension to nonergodic processes. In addition, the assumption of standard spaces simplifies many proofs and such spaces include as examples virtually all examples of engineering interest.

The information convergence results are combined with ergodic theorems to prove general Shannon coding theorems for sources and channels. The results are not the most general known and the converses are not the strongest available in the literature, but they are sufficiently general to cover most sources and single-user communications systems encountered in applications and they are more general than those encountered in most modern texts. For example, most treatments confine interest to stationary and ergodic sources or even independent identically distributed (IID) sources and memoryless channels; here we consider asymptotic mean stationary sources, both one-sided and two-sided sources, and nonergodic sources. General channels with memory are considered, in particular the class of d-bar continuous channels.

Perhaps more important than the generality of the sources and channels is the variety of code structures considered. Most of the literature and virtually all of the texts on information theory focus exclusively on block codes, while many codes are more naturally described as a stationary or sliding-block code — a time-invariant possibly nonlinear filter, generally with a discrete output. Here the basic results of information theory are described for stationary or sliding-block codes as well as for the traditional block codes and the relationships between the two coding structures are explored in detail. Stationary codes arose in ergodic theory in the context of Ornstein's proof of the isomorphism theorem in the 1970s, and they arise naturally in the communications context of classical information theory, including common coding techniques such as time-invariant convolutional codes, predictive quantization, sigma-delta coding, and wavelet transform based techniques that operate as sliding-window or online filters rather than as block operations. Mathematically, stationary codes preserve many of the statistical properties of the source being coded such as stationarity, ergodicity, and mixing. In practice, stationary codes avoid the introduction of blocking artifacts not present in the original source.

This book can be considered as a sequel to my book *Probability, Random Processes, and Ergodic Properties* [58], as the first edition of this book was a sequel to the first edition [56]. There the prerequisite results on probability, standard spaces, and ordinary ergodic properties

may be found along with a development of the general sources considered (asymptotically mean stationary, not necessarily ergodic) and of the process distortion measures used here. This book is self contained with the exception of common (and a few less common) results which may be found in the first book. Results quoted from the first book are cited for both first and second editions as the numbering system in the two editions differs.

It is my hope that the book will interest engineers in some of the mathematical aspects and general models of the theory and mathematicians in some of the important engineering applications of performance bounds and code design for communication systems.

What's New in the Second Edition

As in the second edition of the companion volume [58], material has been corrected, rearranged, and rewritten in an effort to improve the flow of ideas and the presentation. This volume has been revised to reflect the changes in the companion volume, and citations to specific results are given for both the first and second editions [55, 58]. A significant amount of new material has been added both to expand some of the discussions to include more related topics and to include more recent results on old problems.

More general distortion measures are considered when treating the process distance and distortion measures, consistent with extensions or results in [55] on metric distortion measures to powers of metrics (such as the ubiquitous squared-error distortion) in [58].

Three new chapters have been added: one on the interplay between distortion and entropy, one on the interplay between distortion and information, and one on properties of good source codes — codes that are either optimal or asymptotically optimal in the sense of converging to the Shannon limit.

The chapter on distortion and entropy begins with a classic result treated in the first edition, the Fano inequality and its extensions, but it expands the discussion to consider the goodness of approximation of codes and their relation to entropy rate. Pinsker's classic result relating variation distance between probability measures and the divergence (Kullback-Leibler) distance is now treated along with its recent extension by Marton comparing Ornstein's d-bar process distance to divergence rate. The chapter contains a preliminary special case of the coding theorems to come — the application of the entropy ergodic theorem to the design of both block and sliding-block (stationary) almost lossless codes. The example introduces several basic ideas in a relatively simple context, including the construction of a sliding-block code from a block code in a

way that preserves the essential properties. The example also serves to illustrate the connections between information theory and ergodic theory by means of an interpretation of Ornstein's isomorphism theorem — which is not proved here — in terms of almost lossless stationary coding — which is. The results also provide insight into the close relationships between source coding or data compression and rate-constrained simulation of a stationary and ergodic process, the finding of a simple model based on coin flips that resembles as closely as possible the given process.

The chapter on distortion and information adds considerable material on rate-distortion theory to the treatment of the first edition, specifically on the evaluation of Shannon distortion-rate and rate-distortion functions along with their easy applications to lower bounds on performance in idealized communications systems. The fundamentals of Csiszár's variational approach based on the divergence inequality is described and some of the rarely noted attributes are pointed out. The implied algorithm for the evaluation of rate-distortion functions (originally due to Blahut [18]) is interpreted as an early example of alternating optimization.

An entirely new chapter on properties of good codes provides a development along the lines of Gersho and Gray [50] of the basic properties of optimal block codes originally due to Lloyd [110] and Steinhaus [175] along with the implied iterative design algorithm, another early example of alternating optimization. An incomplete extension of these block code optimality properties to sliding-block codes is described, and a simple example of trellis encoding is used to exemplify basic relations between block, sliding-block, and hybrid codes. The remainder of the chapter comprises recent developments in properties of asymptotically optimal sequences of sliding-block codes as developed by Mao, Gray, and Linder [117]. This material adds to the book's emphasis on stationary and sliding-block codes and adds to the limited literature on the subject.

Along with these major additions, I have added many minor results either because I was annoyed to discover they were not already in the first edition when I looked for them or because they eased the development of results.

The addition of three new chapters was partially balanced by the merging of two old chapters to better relate information rates for finite alphabet and continuous alphabet random processes.

Errors

Typographical and technical errors reported to or discovered by me during the two decades since the publication of the first edition have been

corrected and efforts have been made to improve formatting and appearance of the book. Doubtless with the inclusion of new material new errors have occurred. As I age my frequency of typographical and other errors seems to grow along with my ability to see through them. I apologize for any that remain in the book. I will keep a list of all errors found by me or sent to me at rmgray@stanford.edu and I will post the list at my Web site, http://ee.stanford.edu/~gray/.

Acknowledgments

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Introduction

Abstract A brief history of the development of Shannon information theory is presented with an emphasis on its interactions with ergodic theory. The origins and goals of this book are sketched.

Information theory, the mathematical theory of communication, has two primary goals: The first is the development of the fundamental theoretical limits on the achievable performance when communicating a given information source over a given communications channel using coding schemes from within a prescribed class. The second goal is the development of coding schemes that provide performance that is reasonably good in comparison with the optimal performance given by the theory. Information theory was born in a remarkably rich state in the classic papers of Claude E. Shannon [162, 163] which contained the basic results for simple memoryless sources and channels and introduced more general communication systems models, including finite-state sources and channels. The key tools used to prove the original results and many of those that followed were special cases of the ergodic theorem and a new variation of the ergodic theorem which considered sample averages of a measure of the entropy or self information in a process.

Information theory can be viewed as simply a branch of applied probability theory. Because of its dependence on ergodic theorems, however, it can also be viewed as a branch of ergodic theory, the theory of invariant transformations and transformations related to invariant transformations. In order to develop the ergodic theory example of principal interest to information theory, suppose that one has a random process, which for the moment we consider as a sample space or ensemble of possible output sequences together with a probability measure on events

composed of collections of such sequences. The shift is the transformation on this space of sequences that takes a sequence and produces a new sequence by shifting the first sequence a single time unit to the left. In other words, the shift transformation is a mathematical model for the effect of time on a data sequence. If the probability of any sequence event is unchanged by shifting the event, that is, by shifting all of the sequences in the event, then the shift transformation is said to be *invariant* and the random process is said to be *stationary*. Thus the theory of stationary random processes can be considered as a subset of ergodic theory. Transformations that are not actually invariant (random processes which are not actually stationary) can be considered using similar techniques by studying transformations which are almost invariant, which are invariant in an asymptotic sense, or which are dominated or asymptotically dominated in some sense by an invariant transformation. This generality can be important as many real processes are not well modeled as being stationary. Examples are processes with transients, processes that have been parsed into blocks and coded, processes that have been encoded using variable-length codes or finite-state codes, and channels with arbitrary starting states.

Ergodic theory was originally developed for the study of statistical mechanics as a means of quantifying the trajectories of physical or dynamical systems. Hence, in the language of random processes, the early focus was on ergodic theorems: theorems relating the time or sample average behavior of a random process to its ensemble or expected behavior. The work of Hoph [77], von Neumann [190] and others culminated in the pointwise or almost everywhere ergodic theorem of Birkhoff [17].

In the 1940's and 1950's Shannon made use of the ergodic theorem in the simple special case of memoryless processes to characterize the optimal performance possible when communicating an information source over a constrained random medium or *channel* using *codes*. The ergodic theorem was applied in a direct fashion to study the asymptotic behavior of error frequency and time average distortion in a communication system, but a new variation was introduced by defining a mathematical measure of the entropy or information in a random process and characterizing its asymptotic behavior. The results characterizing the optimal performance achievable using codes became known as *coding theorems*. Results describing performance that is actually achievable, at least in the limit of unbounded complexity and time, are known as *positive coding theorems*. Results providing unbeatable bounds on performance are known as *converse coding theorems* or *negative coding theorems.* When the same quantity is given by both positive and negative coding theorems, one has exactly the optimal performance achievable in theory using codes from a given class to communicate through the given communication systems model.

While mathematical notions of information had existed before, it was Shannon who coupled the notion with the ergodic theorem and an ingenious idea known as "random coding" in order to develop the coding theorems and to thereby give operational significance to such information measures. The name "random coding" is a bit misleading since it refers to the random selection of a deterministic code and not a coding system that operates in a random or stochastic manner. The basic approach to proving positive coding theorems was to analyze the average performance over a random selection of codes. If the average is good, then there must be at least one code in the ensemble of codes with performance as good as the average. The ergodic theorem is crucial to this argument for determining such average behavior. Unfortunately, such proofs promise the existence of good codes but give little insight into their construction.

Shannon's original work focused on memoryless sources whose probability distribution did not change with time and whose outputs were drawn from a finite alphabet or the real line. In this simple case the well-known ergodic theorem immediately provided the required result concerning the asymptotic behavior of information. He observed that the basic ideas extended in a relatively straightforward manner to more complicated Markov sources. Even this generalization, however, was a far cry from the general stationary sources considered in the ergodic theorem.

To continue the story requires a few additional words about measures of information. Shannon really made use of two different but related measures. The first was entropy, an idea inherited from thermodynamics and previously proposed as a measure of the information in a random signal by Hartley [75]. Shannon defined the entropy of a discrete time discrete alphabet random process $\{X_n\}$, which we denote by $H(X)$ while deferring its definition, and made rigorous the idea that the the entropy of a process is the amount of information in the process. He did this by proving a coding theorem showing that if one wishes to code the given process into a sequence of binary symbols so that a receiver viewing the binary sequence can reconstruct the original process perfectly (or nearly so), then one needs at least $H(X)$ binary symbols or bits (converse theorem) and one can accomplish the task with very close to $H(X)$ bits (positive theorem). This coding theorem is known as the *noiseless source coding theorem*.

The second notion of information used by Shannon was mutual information. Entropy is really a notion of self information — the information provided by a random process about itself. Mutual information is a measure of the information contained in one process about another process. While entropy is sufficient to study the reproduction of a single process through a noiseless environment, more often one has two or more distinct random processes, e.g., one random process representing an infor-

mation source and another representing the output of a communication medium wherein the coded source has been corrupted by another random process called noise. In such cases observations are made on one process in order to make decisions on another. Suppose that $\{X_n, Y_n\}$ is a random process with a discrete alphabet, that is, taking on values in a discrete set. The coordinate random processes $\{X_n\}$ and $\{Y_n\}$ might correspond, for example, to the input and output of a communication system. Shannon introduced the notion of the average mutual information between the two processes:

$$I(X,Y) = H(X) + H(Y) - H(X,Y), \tag{1}$$

the sum of the two self entropies minus the entropy of the pair. This proved to be the relevant quantity in coding theorems involving more than one distinct random process: the channel coding theorem describing reliable communication through a noisy channel, and the general source coding theorem describing the coding of a source for a user subject to a fidelity criterion. The first theorem focuses on error detection and correction and the second on analog-to-digital conversion and data compression. Special cases of both of these coding theorems were given in Shannon's original work.

Average mutual information can also be defined in terms of *conditional entropy* $H(X|Y) = H(X,Y) - H(Y)$ and hence

$$I(X,Y) = H(X) - H(X|Y) = H(Y) - H(X|Y). \tag{2}$$

In this form the mutual information can be interpreted as the information contained in one process minus the information contained in the process when the other process is known. While elementary texts on information theory abound with such intuitive descriptions of information measures, we will minimize such discussion because of the potential pitfall of using the interpretations to apply such measures to problems where they are not appropriate. (See, e.g., P. Elias' "Information theory, photosynthesis, and religion" in his "Two famous papers" [37].) Information measures are important because coding theorems exist imbuing them with operational significance and not because of intuitively pleasing aspects of their definitions.

We focus on the definition (1) of mutual information since it does not require any explanation of what conditional entropy means and since it has a more symmetric form than the conditional definitions. It turns out that $H(X,X) = H(X)$ (the entropy of a random variable is not changed by repeating it) and hence from (1)

$$I(X,X) = H(X) \tag{3}$$

so that entropy can be considered as a special case of average mutual information.

To return to the story, Shannon's work spawned the new field of information theory and also had a profound effect on the older field of ergodic theory.

Information theorists, both mathematicians and engineers, extended Shannon's basic approach to ever more general models of information sources, coding structures, and performance measures. The fundamental ergodic theorem for entropy was extended to the same generality as the ordinary ergodic theorems by McMillan [123] and Breiman [20] and the result is now known as the Shannon-McMillan-Breiman theorem. Other names are the asymptotic equipartition theorem or AEP, the ergodic theorem of information theory, and the entropy theorem. A variety of detailed proofs of the basic coding theorems and stronger versions of the theorems for memoryless, Markov, and other special cases of random processes were developed, notable examples being the work of Feinstein [39] [40] and Wolfowitz (see, e.g., Wolfowitz [196].) The ideas of measures of information, channels, codes, and communications systems were rigorously extended to more general random processes with abstract alphabets and discrete and continuous time by Khinchine [87], [88] and by Kolmogorov and his colleagues, especially Gelfand, Yaglom, Dobrushin, and Pinsker [49], [104], [101], [32], [150]. (See, for example, "Kolmogorov's contributions to information theory and algorithmic complexity" [23].) In almost all of the early Soviet work, it was average mutual information that played the fundamental role. It was the more natural quantity when more than one process were being considered. In addition, the notion of entropy was not useful when dealing with processes with continuous alphabets since it is generally infinite in such cases. A generalization of the idea of entropy called *discrimination* was developed by Kullback (see, e.g., Kullback [106]) and was further studied by the Soviet school. This form of information measure is now more commonly referred to as relative entropy, cross entropy, or Kullback-Leibler number, or information divergence and it is better interpreted as a measure of similarity or dissimilarity between probability distributions than as a measure of information between random variables. Many results for mutual information and entropy can be viewed as special cases of results for relative entropy and the formula for relative entropy arises naturally in some proofs.

It is the mathematical aspects of information theory and hence the descendants of the above results that are the focus of this book, but the developments in the engineering community have had as significant an impact on the foundations of information theory as they have had on applications. Simpler proofs of the basic coding theorems were developed for special cases and, as a natural offshoot, the rate of convergence to the optimal performance bounds characterized in a variety of important

cases. See, e.g., the texts by Gallager [47], Berger [11], and Csiszàr and Körner [27]. Numerous practicable coding techniques were developed which provided performance reasonably close to the optimum in many cases: from the simple linear error correcting and detecting codes of Slepian [171] to the huge variety of algebraic codes that have been implemented (see, e.g., [12], [192],[109], [113], [19]), the various forms of convolutional, tree, and trellis codes for error correction and data compression (see, e.g., [189, 81]), and the recent codes approaching the Shannon limits based on iterative coding and message passage ideas [126, 156], codes which have their roots in Gallager's PhD thesis on low density parity check codes [48]. Codes for source coding and data compression include a variety of traditional and recent techniques for lossless coding of data and lossy coding of realtime signals such as voice, audio, still images, and video. Techniques range from simple quantization to predictive quantization, adaptive methods, vector quantizers based on linear transforms followed by quantization and lossless codes, subband coders, and model coders such as the linear preditive codes for voice which fit linear models to observed signals for local synthesis. A sampling of the fundamentals through the standards can be found in [50, 160, 144, 178].

The engineering side of information theory through the middle 1970's has been well chronicled by two IEEE collections: *Key Papers in the Development of Information Theory,* edited by D. Slepian [172], and *Key Papers in the Development of Coding Theory,* edited by E. Berlekamp [13] and many papers describing the first fifty years of the field were collected into *Information Theory: 50 Years of Discovery* in 2000 [184]. In addition there have been several survey papers describing the history of information theory during each decade of its existence published in the *IEEE Transactions on Information Theory.*

The influence on ergodic theory of Shannon's work was equally great but in a different direction. After the development of quite general ergodic theorems, one of the principal issues of ergodic theory was the isomorphism problem, the characterization of conditions under which two dynamical systems are really the same in the sense that each could be obtained from the other in an invertible way by coding. Here, however, the coding was not of the variety considered by Shannon — Shannon considered block codes, codes that parsed the data into nonoverlapping blocks or windows of finite length and separately mapped each input block into an output block. The more natural construct in ergodic theory can be called a sliding-block code or stationary code — here the encoder views a block of possibly infinite length and produces a single symbol of the output sequence using some mapping (or code or filter). The input sequence is then shifted one time unit to the left, and the same mapping applied to produce the next output symbol, and so on. This is a smoother operation than the block coding structure since the outputs

are produced based on overlapping windows of data instead of on a completely different set of data each time. Unlike the Shannon codes, these codes will produce stationary output processes if given stationary input processes. It should be mentioned that examples of such sliding-block codes often occurred in the information theory literature: time-invariant convolutional codes or, simply, time-invariant linear filters are sliding-block codes. It is perhaps odd that virtually all of the theory for such codes in the information theory literature was developed by effectively considering the sliding-block codes as very long block codes. Sliding-block codes have proved a useful structure for the design of noiseless codes for constrained alphabet channels such as magnetic recording devices, and techniques from symbolic dynamics have been applied to the design of such codes. See, for example [3, 118].

Shannon's noiseless source coding theorem suggested a solution to the isomorphism problem: If we assume for the moment that one of the two processes is binary, then perfect coding of a process into a binary process and back into the original process requires that the original process and the binary process have the same entropy. Thus a natural conjecture is that two processes are isomorphic if and only if they have the same entropy. A major difficulty was the fact that two different kinds of coding were being considered: stationary sliding-block codes with zero error by the ergodic theorists and either fixed length block codes with small error or variable length (and hence nonstationary) block codes with zero error by the Shannon theorists. While it was plausible that the former codes might be developed as some sort of limit of the latter, this proved to be an extremely difficult problem. It was Kolmogorov [102], [103] who first reasoned along these lines and proved that in fact equal entropy (appropriately defined) was a necessary condition for isomorphism.

Kolmogorov's seminal work initiated a new branch of ergodic theory devoted to the study of entropy of dynamical systems and its application to the isomorphism problem. Most of the original work was done by Soviet mathematicians; notable papers are those by Sinai [168] [169] (in ergodic theory entropy is also known as the Kolmogorov-Sinai invariant), Pinsker [150], and Rohlin and Sinai [157]. An actual construction of a perfectly noiseless sliding-block code for a special case was provided by Meshalkin [124]. While much insight was gained into the behavior of entropy and progress was made on several simplified versions of the isomorphism problem, it was several years before Ornstein [138] proved a result that has since come to be known as the Ornstein isomorphism theorem or the Kolmogorov-Ornstein or Kolmogorov-Sinai-Ornstein isomorphism theorem.

Ornstein showed that if one focused on a class of random processes which we shall call B-processes, then two processes are indeed isomorphic if and only if they have the same entropy. *B*-process are also called

Bernoulli processes in the ergodic theory literature, but this is potentially confusing because of the usage of "Bernoulli process" as a synonym of an independent identically distributed (IID) process in information theory and random process theory. B-processes have several equivalent definitions, perhaps the simplest is that they are processes which can be obtained by encoding a memoryless process using a sliding-block code. This class remains the most general class known for which the isomorphism conjecture holds. In the course of his proof, Ornstein developed intricate connections between block coding and sliding-block coding. He used Shannon-like techniques on the block codes, then imbedded the block codes into sliding-block codes, and then used the stationary structure of the sliding-block codes to advantage in limiting arguments to obtain the required zero error codes. Several other useful techniques and results were introduced in the proof: notions of the distance between processes and relations between the goodness of approximation and the difference of entropy. Ornstein expanded these results into a book [140] and gave a tutorial discussion in the premier issue of the *Annals of Probability* [139]. Several correspondence items by other ergodic theorists discussing the paper accompanied the article.

The origins of this book lie in the tools developed by Ornstein for the proof of the isomorphism theorem rather than with the result itself. During the early 1970's I first become interested in ergodic theory because of joint work with Lee D. Davisson on source coding theorems for stationary nonergodic processes. The ergodic decomposition theorem discussed in Ornstein [139] provided a needed missing link and led to an intense campaign on my part to learn the fundamentals of ergodic theory and perhaps find other useful tools. This effort was greatly eased by Paul Shields' book *The Theory of Bernoulli Shifts* [164] and by discussions with Paul on topics in both ergodic theory and information theory. This in turn led to a variety of other applications of ergodic theoretic techniques and results to information theory, mostly in the area of source coding theory: proving source coding theorems for sliding-block codes and using process distance measures to prove universal source coding theorems and to provide new characterizations of Shannon distortion-rate functions. The work was done with Dave Neuhoff, like me then an apprentice ergodic theorist, and Paul Shields.

With the departure of Dave and Paul from Stanford, my increasing interest led me to discussions with Don Ornstein on possible applications of his techniques to channel coding problems. The interchange often consisted of my describing a problem, his generation of possible avenues of solution, and then my going off to work for a few weeks to understand his suggestions and work them through.

One problem resisted our best efforts–how to synchronize block codes over channels with memory, a prerequisite for constructing sliding-block codes for such channels. In 1975 I had the good fortune to meet and talk

with Roland Dobrushin at the 1975 IEEE/USSR Workshop on Information Theory in Moscow. He observed that some of his techniques for handling synchronization in memoryless channels should immediately generalize to our case and therefore should provide the missing link. The key elements were all there, but it took seven years for the paper by Ornstein, Dobrushin and me to evolve and appear [68].

Early in the course of the channel coding paper, I decided that having the solution to the sliding-block channel coding result in sight was sufficient excuse to write a book on the overlap of ergodic theory and information theory. The intent was to develop the tools of ergodic theory of potential use to information theory and to demonstrate their use by proving Shannon coding theorems for the most general known information sources, channels, and code structures. Progress on the book was disappointingly slow, however, for a number of reasons. As delays mounted, I saw many of the general coding theorems extended and improved by others (often by J. C. Kieffer) and new applications of ergodic theory to information theory developed, such as the channel modeling work of Neuhoff and Shields [133], [136], [135], [134] and design methods for sliding-block codes for input restricted noiseless channels by Adler, Coppersmith, and Hasner [3] and Marcus [118]. Although I continued to work in some aspects of the area, especially with nonstationary and nonergodic processes and processes with standard alphabets, the area remained for me a relatively minor one and I had little time to write. Work and writing came in bursts during sabbaticals and occasional advanced topic seminars. I abandoned the idea of providing the most general possible coding theorems and decided instead to settle for coding theorems that were sufficiently general to cover most applications and which possessed proofs I liked and could understand.

Only one third of this book is actually devoted to Shannon source and channel coding theorems; the remainder can be viewed as a monograph on sources, channels, and codes and on information and distortion measures and their properties, especially their ergodic properties. The sources or random processes considered include asymptotically mean stationary processes with standard alphabets, a subject developed in detail in my earlier book *Probability, Random Processes, and Ergodic Properties*, which was published by Springer-Verlag in 1988 [55] with a second edition published by Springer in 2009. That books treats advanced probability and random processes with an emphasis on processes with standard alphabets, on nonergodic and nonstationary processes, and on necessary and sufficient conditions for the convergence of long term sample averages. Asymptotically mean stationary sources and the ergodic decomposition are there treated in depth and recent simplified proofs of the ergodic theorem due to Ornstein and Weiss [141] and others are incorporated. The next chapter of this book reviews some of the basic notation of the first one in information theoretic terms, but results are

often simply quoted as needed from the first book without any attempt to derive them. The two books together are self-contained in that all supporting results from probability theory and ergodic theory needed here may be found in the first book. This book is self-contained so far as its information theory content, but it should be considered as an advanced text on the subject and not as an introductory treatise to the reader only wishing an intuitive overview. The border between the two books is the beginning of the treatment of entropy.

Here the Shannon-McMillan-Breiman theorem is proved using the coding approach of Ornstein and Weiss [141] (see also Shield's tutorial paper [165]) and hence the treatments of ordinary ergodic theorems in the first book and the ergodic theorems for information measures in this book are consistent. The extension of the Shannon-McMillan-Breiman theorem to densities is proved using the "sandwich" approach of Algoet and Cover [7], which depends strongly on the usual pointwise or Birkhoff ergodic theorem: sample entropy is asymptotically sandwiched between two functions whose limits can be determined from the ergodic theorem. These results are the most general yet published in book form and differ from traditional developments in that martingale theory is not required in the proofs.

A few words are in order regarding topics that are not contained in this book. I have not included the increasingly important and growing area of multiuser information theory because my experience in the area is slight and I believe this topic can be better handled by others.

Traditional noiseless coding theorems and actual codes such as the Huffman codes are not considered in depth because quite good treatments exist in the literature, e.g., [47], [1], [122]. The corresponding ergodic theory result — the Ornstein isomorphism theorem — is also not proved, because its proof is difficult and the result is not needed for the Shannon coding theorems. It is, however, described and many techniques used in its proof are used here for similar and other purposes.

The actual computation of channel capacity and distortion rate functions has not been included because existing treatments [47], [18], [11], [25] [57] are quite adequate. New to the second edition, however, is a partial development of Csiszár's [25] rigorous development of the information-theoretic optimization underlying the evaluation of the rate-distortion function.

This book does not treat code design techniques in any depth, but in this second edition properties of optimal and asymptotically optimal source codes are developed and these properties provide insight into the structure of good codes and can be used to guide code design. The traditional Lloyd optimality properties for vector quantizers are described along with recent results for sliding-block codes which resemble their block coding cousins.

J. C. Kieffer developed a powerful new ergodic theorem that can be used to prove both traditional ergodic theorems and the extended Shannon-McMillan-Brieman theorem [96]. He has used this theorem to prove strong (almost everywhere) versions of the source coding theorem and its converse, that is, results showing that sample average distortion is with probability one no smaller than the distortion-rate function and that there exist codes with sample average distortion arbitrarily close to the distortion-rate function [99, 100].

Chapter 1
Information Sources

Abstract An *information source* or *source* is a mathematical model for a physical entity that produces a succession of symbols called "outputs" in a random manner. The symbols produced may be real numbers such as voltage measurements from a transducer, binary numbers as in computer data, two dimensional intensity fields as in a sequence of images, continuous or discontinuous waveforms, and so on. The space containing all of the possible output symbols is called the *alphabet* of the source and a source is essentially an assignment of a probability measure to events consisting of sets of sequences of symbols from the alphabet. It is useful, however, to explicitly treat the notion of time as a transformation of sequences produced by the source. Thus in addition to the common random process model we shall also consider modeling sources by dynamical systems as considered in ergodic theory. The material in this chapter is a distillation of [55, 58] and is intended to establish notation.

1.1 Probability Spaces and Random Variables

A measurable space $(\Omega, \mathcal{B})$ is a pair consisting of a sample space Ω together with a σ-field $\mathcal{B}$ of subsets of Ω (also called the event space). A σ-field or σ-algebra $\mathcal{B}$ is a nonempty collection of subsets of Ω with the following properties:

$$\Omega \in \mathcal{B}. \tag{1.1}$$

$$\text{If } F \in \mathcal{B}, \text{ then } F^c = \{\omega : \omega \notin F\} \in \mathcal{B}. \tag{1.2}$$

$$\text{If } F_i \in \mathcal{B}; i = 1, 2, \ldots, \text{ then } \bigcup_i F_i \in \mathcal{B}. \tag{1.3}$$

From de Morgan's "laws" of elementary set theory it follows that also

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$$\bigcap_{i=1}^{\infty} F_i = (\bigcup_{i=1}^{\infty} F_i^c)^c \in \mathcal{B}.$$

An event space is a collection of subsets of a sample space (called events by virtue of belonging to the event space) such that any countable sequence of set theoretic operations (union, intersection, complementation) on events produces other events. Note that there are two extremes: the largest possible σ-field of Ω is the collection of all subsets of Ω (sometimes called the *power set*) and the smallest possible σ-field is $\{\Omega, \varnothing\}$, the entire space together with the null set $\varnothing = \Omega^c$ (called the *trivial space*).

If instead of the closure under countable unions required by (1.3), we only require that the collection of subsets be closed under finite unions, then we say that the collection of subsets is a *field.*

While the concept of a field is simpler to work with, a σ-field possesses the additional important property that it contains all of the limits of sequences of sets in the collection. That is, if F_n, $n = 1, 2, \cdots$ is an increasing sequence of sets in a σ-field, that is, if $F_{n-1} \subset F_n$ and if $F = \bigcup_{n=1}^{\infty} F_n$ (in which case we write $F_n \uparrow F$ or $\lim_{n\to\infty} F_n = F$), then also F is contained in the σ-field. In a similar fashion we can define decreasing sequences of sets: If F_n decreases to F in the sense that $F_{n+1} \subset F_n$ and $F = \bigcap_{n=1}^{\infty} F_n$, then we write $F_n \downarrow F$. If $F_n \in \mathcal{B}$ for all n, then $F \in \mathcal{B}$.

A *probability space* $(\Omega, \mathcal{B}, P)$ is a triple consisting of a sample space Ω , a σ-field $\mathcal{B}$ of subsets of Ω , and a probability measure P which assigns a real number $P(F)$ to every member F of the σ-field $\mathcal{B}$ so that the following conditions are satisfied:

- *Nonnegativity*:
$$P(F) \geq 0, \text{ all } F \in \mathcal{B}; \tag{1.4}$$
- *Normalization*:
$$P(\Omega) = 1; \tag{1.5}$$
- *Countable Additivity*:

If $F_i \in \mathcal{B}, i = 1, 2, \cdots$ are disjoint, then

$$P(\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} P(F_i). \tag{1.6}$$

A set function P satisfying only (1.4) and (1.6) but not necessarily (1.5) is called a *measure* and the triple $(\Omega, \mathcal{B}, P)$ is called a *measure space*. Since the probability measure is defined on a σ-field, such countable unions of subsets of Ω in the σ-field are also events in the σ-field.

A standard result of basic probability theory is that if $G_n \downarrow \varnothing$ (the empty or null set), that is, if $G_{n+1} \subset G_n$ for all n and $\bigcap_{n=1}^{\infty} G_n = \varnothing$, then we have

- *Continuity at* $\emptyset$:

$$\lim_{n\to\infty} P(G_n) = 0. \tag{1.7}$$

similarly it follows that we have

- *Continuity from Below*:

$$\text{If } F_n \uparrow F, \text{ then } \lim_{n\to\infty} P(F_n) = P(F), \tag{1.8}$$

and

- *Continuity from Above*:

$$\text{If } F_n \downarrow F, \text{ then } \lim_{n\to\infty} P(F_n) = P(F). \tag{1.9}$$

Given a measurable space $(\Omega, \mathcal{B})$, a collection $\mathcal{G}$ of members of $\mathcal{B}$ is said to *generate* $\mathcal{B}$ and we write $\sigma(\mathcal{G}) = \mathcal{B}$ if $\mathcal{B}$ is the smallest σ-field that contains $\mathcal{G}$; that is, if a σ-field contains all of the members of $\mathcal{G}$, then it must also contain all of the members of $\mathcal{B}$. The following is a fundamental approximation theorem of probability theory. A proof may be found in Corollary 1.5.3 of [55] or Corollary 1.5 of [58]. The result is most easily stated in terms of the symmetric difference Δ defined by

$$F\Delta G \equiv (F \cap G^c) \cup (F^c \cap G).$$

Theorem 1.1. *Given a probability space* $(\Omega, \mathcal{B}, P)$ *and a generating field* $\mathcal{F}$*, that is,* $\mathcal{F}$ *is a field and* $\mathcal{B} = \sigma(\mathcal{F})$*, then given* $F \in \mathcal{B}$ *and* $\epsilon > 0$*, there exists an* $F_0 \in \mathcal{F}$ *such that* $P(F\Delta F_0) \le \epsilon$*.*

Let $(A, \mathcal{B}_A)$ denote another measurable space. We will also use $\mathcal{B}(A)$ as a synonym for $\mathcal{B}_A$. A *random variable* or *measurable function* defined on $(\Omega, \mathcal{B})$ and taking values in (A,$\mathcal{B}_A$) is a mapping or function $f : \Omega \to A$ with the property that

$$\text{if } F \in \mathcal{B}_A, \text{ then } f^{-1}(F) = \{\omega : f(\omega) \in F\} \in \mathcal{B}. \tag{1.10}$$

The name "random variable" is commonly associated with the special case where A is the real line and $\mathcal{B}$ the Borel field, the smallest σ-field containing all the intervals. Occasionally a more general sounding name such as "random object" is used for a measurable function to implicitly include random variables (A the real line), random vectors (A a Euclidean space), and random processes (A a sequence or waveform space). We will use the terms "random variable" in the more general sense. Usually A will either be a metric space or a product of metric spaces, in which case the σ-field will be a Borel field $\mathcal{B}_A$ or $\mathcal{B}(A)$ of subsets of A. If A is a product of metric spaces, then $\mathcal{B}_A$ will be taken as the corresponding product σ-field, that is, the σ-field generated by the rectangles.

A random variable is just a function or mapping with the property that inverse images of "output events" determined by the random variable are events in the original measurable space. This simple property ensures that the output of the random variable will inherit its own probability measure. For example, with the probability measure P_f defined by

$$P_f(B) = P(f^{-1}(B)) = P(\omega : f(\omega) \in B);\ B \in \mathcal{B}_A,$$

$(A, \mathcal{B}_A, P_f)$ becomes a probability space since measurability of f and elementary set theory ensure that P_f is indeed a probability measure. The induced probability measure P_f is called the *distribution* of the random variable f. The measurable space $(A, \mathcal{B}_A)$ or, simply, the sample space A, is called the alphabet of the random variable f. We shall occasionally also use the notation Pf^{-1} which is a mnemonic for the relation $Pf^{-1}(F) = P(f^{-1}(F))$ and which is less awkward when f itself is a function with a complicated name, e.g., $\Pi_{\mathcal{I}\to\mathcal{M}}$.

It is often convenient to abbreviate an English description the of a probability of an event to the pseudo mathematical form $\Pr(f \in F)$, which can be considered shorthand for $P_f(F) = P(f^{-1}(F))$ and can be read as "the probability that f is in F."

If the alphabet A of a random variable f is not clear from context, then we shall refer to f as an *A-valued random variable.* . If f is a measurable function from $(\Omega, \mathcal{B})$ to $(A, \mathcal{B}_A)$, we will say that f is $\mathcal{B}/\mathcal{B}_A$-measurable if the σ-fields might not be clear from context.

Given a probability space $(\Omega, \mathcal{B}, P)$, a collection of subsets $\mathcal{G}$ is a sub-σ-field if it is a σ-field and all its members are in $\mathcal{B}$. A random variable $f : \Omega \to A$ is said to be measurable with respect to a sub-σ-field $\mathcal{G}$ if $f^{-1}(H) \in \mathcal{G}$ for all $H \in \mathcal{B}_A$.

Given a probability space $(\Omega, \mathcal{B}, P)$ and a sub-σ-field $\mathcal{G}$, for any event $H \in \mathcal{B}$ the conditional probability $m(H|\mathcal{G})$ is defined as any function, say g, which satisfies the two properties

$$g \text{ is measurable with respect to } \mathcal{G} \tag{1.11}$$

$$\int_G g h dP = m(G \bigcap H);\ \text{ all } G \in \mathcal{G}. \tag{1.12}$$

An important special case of conditional probability occurs when studying the distributions of random variables defined on an underlying probability space. Suppose that $X : \Omega \to A_X$ and $Y : \Omega \to A_Y$ are two random variables defined on $(\Omega, \mathcal{B}, P)$ with alphabets A_X and A_Y and σ-fields $\mathcal{B}_{A_X}$ and $\mathcal{B}_{A_Y}$, respectively. Let P_{XY} denote the induced distribution on $(A_X \times A_Y, \mathcal{B}_{A_X} \times \mathcal{B}_{A_Y})$, that is, $P_{XY}(F \times G) = P(X \in F, Y \in G) = P(X^{-1}(F) \bigcap Y^{-1}(G))$. Let $\sigma(Y)$ denote the sub-σ-field of $\mathcal{B}$ generated by Y, that is, $Y^{-1}(\mathcal{B}_{A_Y})$. Since the conditional probability $P(F|\sigma(Y))$ is real-valued and measurable with respect to $\sigma(Y)$, it can be written as

$g(Y(\omega))$, $\omega \in \Omega$, for some function $g(y)$. (See, for example, Lemma 5.2.1 of [55] or Lemma 6.1 of [58].) Define $P(F|y) = g(y)$. For a fixed $F \in \mathcal{B}_{A_X}$ define the *conditional distribution* of F given $Y = y$ by

$$P_{X|Y}(F|y) = P(X^{-1}(F)|y);\ y \in \mathcal{B}_{A_Y}.$$

From the properties of conditional probability,

$$P_{XY}(F \times G) = \int_G P_{X|Y}(F|y)dP_Y(y); F \in \mathcal{B}_{A_X}, G \in \mathcal{B}_{A_Y}. \tag{1.13}$$

It is tempting to think that for a fixed y, the set function defined by $P_{X|Y}(F|y); F \in \mathcal{B}_{A_X}$ is actually a probability measure. This is not the case in general. When it does hold for a conditional probability measure, the conditional probability measure is said to be *regular*. This text will focus on standard alphabets for which regular conditional probabilities always exist.

1.2 Random Processes and Dynamical Systems

We now consider two mathematical models for a source: A random process and a dynamical system. The first is the familiar one in elementary courses, a source is just a random process or sequence of random variables. The second model is possibly less familiar — a random process can also be constructed from an abstract dynamical system consisting of a probability space together with a transformation on the space. The two models are connected by considering a time shift to be a transformation.

A *discrete time random process* or, simply, a *random process* is a sequence of random variables $\{X_n\}_{n\in\mathbb{T}}$ or $\{X_n; n \in \mathbb{T}\}$, where $\mathbb{T}$ is an index set, defined on a common probability space $(\Omega, \mathcal{B}, P)$. We define a *source* as a random process, although we could also use the alternative definition of a dynamical system to be introduced shortly. We usually assume that all of the random variables share a common alphabet, say A. The two most common index sets of interest are the set of all integers $\mathcal{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$, in which case the random process is referred to as a *two-sided* random process, and the set of all nonnegative integers $\mathcal{Z}_+ = \{0, 1, 2, \cdots\}$, in which case the random process is said to be *one-sided*. One-sided random processes will often prove to be far more difficult in theory, but they provide better models for physical random processes that must be "turned on" at some time or which have transient behavior.

Observe that since the alphabet A is general, we could also model continuous time random processes in the above fashion by letting A

consist of a family of waveforms defined on an interval, e.g., the random variable X_n could in fact be a continuous time waveform $X(t)$ for $t \in [nT, (n+1)T)$, where T is some fixed positive real number.

The above definition does not specify any structural properties of the index set $\mathbb{T}$. In particular, it does not exclude the possibility that $\mathbb{T}$ be a finite set, in which case "random vector" would be a better name than "random process." In fact, the two cases of $\mathbb{T} = \mathcal{Z}$ and $\mathbb{T} = \mathcal{Z}_+$ will be the only important examples for our purposes. Nonetheless, the general notation of $\mathbb{T}$ will be retained in order to avoid having to state separate results for these two cases.

An abstract dynamical system consists of a probability space $(\Omega, \mathcal{B}, P)$ together with a measurable transformation $T : \Omega \to \Omega$ of Ω into itself. Measurability means that if $F \in \mathcal{B}$, then also $T^{-1}F = \{\omega : T\omega \in F\} \in \mathcal{B}$. The quadruple $(\Omega,\mathcal{B},P,T)$ is called a *dynamical system* in ergodic theory. The interested reader can find excellent introductions to classical ergodic theory and dynamical system theory in the books of Halmos [73] and Sinai [170]. More complete treatments may be found in [16], [164], [149], [30], [191], [140], [46]. The term "dynamical systems" comes from the focus of the theory on the long term "dynamics" or "dynamical behavior" of repeated applications of the transformation T on the underlying measure space.

An alternative to modeling a random process as a sequence or family of random variables defined on a common probability space is to consider a single random variable together with a transformation defined on the underlying probability space. The outputs of the random process will then be values of the random variable taken on transformed points in the original space. The transformation will usually be related to shifting in time and hence this viewpoint will focus on the action of time itself. Suppose now that T is a measurable mapping of points of the sample space Ω into itself. It is easy to see that the cascade or composition of measurable functions is also measurable. Hence the transformation T^n defined as $T^2\omega = T(T\omega)$ and so on ($T^n\omega = T(T^{n-1}\omega)$) is a measurable function for all positive integers n. If f is an A-valued random variable defined on $(\Omega, \mathcal{B})$, then the functions $fT^n : \Omega \to A$ defined by $fT^n(\omega) = f(T^n\omega)$ for $\omega \in \Omega$ will also be random variables for all n in $\mathcal{Z}_+$. Thus a dynamical system together with a random variable or measurable function f defines a one-sided random process $\{X_n\}_{n\in\mathcal{Z}_+}$ by $X_n(\omega) = f(T^n\omega)$. If it should be true that T is invertible, that is, T is one-to-one and its inverse T^{-1} is measurable, then one can define a two-sided random process by $X_n(\omega) = f(T^n\omega)$, all n in $\mathcal{Z}$.

The most common dynamical system for modeling random processes is that consisting of a sequence space Ω containing all one- or two-sided A-valued sequences together with the shift transformation T, that is, the transformation that maps a sequence $\{x_n\}$ into the sequence $\{x_{n+1}\}$ wherein each coordinate has been shifted to the left by one time unit.

Thus, for example, let $\Omega = A^{\mathcal{Z}_+} = \{\text{all } x = (x_0, x_1, \cdots) \text{ with } x_i \in A \text{ for all } i\}$ and define $T : \Omega \to \Omega$ by $T(x_0, x_1, x_2, \cdots) = (x_1, x_2, x_3, \cdots)$. T is called the *shift* or *left shift* transformation on the one-sided sequence space. The shift for two-sided spaces is defined similarly. The sequence-space model of a random process is sometimes referred to as the Kolmogorov representation of a process.

The different models provide equivalent models for a given process — one emphasizing the sequence of outputs and the other emphasising the action of a transformation on the underlying space in producing these outputs. In order to demonstrate in what sense the models are equivalent for given random processes, we next turn to the notion of the distribution of a random process.

1.3 Distributions

While in principle all probabilistic quantities associated with a random process can be determined from the underlying probability space, it is often more convenient to deal with the induced probability measures or distributions on the space of possible outputs of the random process. In particular, this allows us to compare different random processes without regard to the underlying probability spaces and thereby permits us to reasonably equate two random processes if their outputs have the same probabilistic structure, even if the underlying probability spaces are quite different.

We have already seen that each random variable X_n of the random process $\{X_n\}$ inherits a distribution because it is measurable. To describe a process, however, we need more than just probability measures on output values of separate individual random variables; we require probability measures on collections of random variables, that is, on sequences of outputs. In order to place probability measures on sequences of outputs of a random process, we first must construct the appropriate measurable spaces. A convenient technique for accomplishing this is to consider product spaces, spaces for sequences formed by concatenating spaces for individual outputs.

Let $\mathbb{T}$ denote any finite or infinite set of integers. In particular, $\mathbb{T} = \mathcal{Z}(n) = \{0, 1, 2, \cdots, n-1\}$, $\mathbb{T} = \mathcal{Z}$, or $\mathbb{T} = \mathcal{Z}_+$. Define $x^{\mathbb{T}} = \{x_i\}_{i \in \mathbb{T}}$. For example, $x^{\mathcal{Z}} = (\cdots, x_{-1}, x_0, x_1, \cdots)$ is a two-sided infinite sequence. When $\mathbb{T} = \mathcal{Z}(n)$ we abbreviate $x^{\mathcal{Z}(n)}$ to simply x^n . Given alphabets $A_i, i \in \mathbb{T}$, define the cartesian product space

$$\underset{i \in \mathbb{T}}{\times} A_i = \{\text{all } x^{\mathbb{T}} : x_i, \in A_i \text{ all } i \text{ in } \mathbb{T}\}.$$

In most cases all of the A_i will be replicas of a single alphabet A and the above product will be denoted simply by $A^{\mathbb{T}}$. Thus, for example, $A^{\{m,m+1,\cdots,n\}}$ is the space of all possible outputs of the process from time m to time n; $A^{\mathcal{Z}}$ is the sequence space of all possible outputs of a two-sided process. We shall abbreviate the notation for the space $A^{\mathcal{Z}(n)}$, the space of all n dimensional vectors with coordinates in A, by A^n .

To obtain useful σ-fields of the above product spaces, we introduce the idea of a rectangle in a product space. A *rectangle* in $A^{\mathbb{T}}$ taking values in the coordinate σ-fields $\mathcal{B}_i,\ i \in \mathbb{J}$, is defined as any set of the form

$$B = \{x^{\mathbb{T}} \in A^{\mathbb{T}} :\ x_i \in B_i;\ \text{all } i \text{ in } \mathbb{J}\}, \tag{1.14}$$

where $\mathbb{J}$ is a finite subset of the index set $\mathbb{T}$ and $B_i \in \mathcal{B}_i$ for all $i \in \mathbb{J}$. (Hence rectangles are sometimes referred to as finite dimensional rectangles.) A rectangle as in (1.14) can be written as a finite intersection of one-dimensional rectangles as

$$B = \bigcap_{i\in\mathbb{J}} \{x^{\mathbb{T}} \in A^{\mathbb{T}} : x_i \in B_i\} = \bigcap_{i\in\mathbb{J}} X_i^{-1}(B_i) \tag{1.15}$$

where here we consider X_i as the coordinate functions $X_i : A^{\mathbb{T}} \to A$ defined by $X_i(x^{\mathbb{T}}) = x_i$.

As rectangles in $A^{\mathbb{T}}$ are clearly fundamental events, they should be members of any useful σ-field of subsets of $A^{\mathbb{T}}$. Define the product σ-field $\mathcal{B}_A{}^{\mathbb{T}}$ as the smallest σ-field containing all of the rectangles, that is, the collection of sets that contains the clearly important class of rectangles and the minimum amount of other stuff required to make the collection a σ-field. To be more precise, given an index set $\mathbb{T}$ of integers, let $RECT(\mathcal{B}_i, i \in \mathbb{T})$ denote the set of all rectangles in $A^{\mathbb{T}}$ taking coordinate values in sets in $\mathcal{B}_i, i \in \mathbb{T}$. We then define the product σ-field of $A^{\mathbb{T}}$ by

$$\mathcal{B}_A{}^{\mathbb{T}} = \sigma(RECT(\mathcal{B}_i, i \in \mathbb{T})). \tag{1.16}$$

Consider an index set $\mathbb{T}$ and an A-valued random process $\{X_n\}_{n\in\mathbb{T}}$ defined on an underlying probability space $(\Omega, \mathcal{B}, P)$. Given any index set $\mathbb{J} \subset \mathbb{T}$, measurability of the individual random variables X_n implies that of the random vectors $X^{\mathbb{J}} = \{X_n; n \in \mathbb{J}\}$. Thus the measurable space $(A^{\mathbb{J}}, \mathcal{B}_A{}^{\mathbb{J}})$ inherits a probability measure from the underlying space through the random variables $X^{\mathbb{J}}$. Thus in particular the measurable space $(A^{\mathbb{T}}, \mathcal{B}_A{}^{\mathbb{T}})$ inherits a probability measure from the underlying probability space and thereby determines a new probability space $(A^{\mathbb{T}}, \mathcal{B}_A{}^{\mathbb{T}}, P_{X^{\mathbb{T}}})$, where the induced probability measure is defined by

$$P_{X^{\mathbb{T}}}(F) = P((X^{\mathbb{T}})^{-1}(F)) = P(\omega :\ X^{\mathbb{T}}(\omega) \in F);\ F \in \mathcal{B}_A{}^{\mathbb{T}}. \tag{1.17}$$

Such probability measures induced on the outputs of random variables are referred to as *distributions* for the random variables, exactly as in the simpler case first treated. When $\mathbb{T} = \{m, m+1, \cdots, m+n-1\}$, e.g., when we are treating $X_m^n = (X_n, \cdots, X_{m+n-1})$ taking values in A^n, the distribution is referred to as an n-dimensional or nth order distribution and it describes the behavior of an n-dimensional random variable. If $\mathbb{T}$ is the entire process index set, e.g., if $\mathbb{T} = \mathcal{Z}$ for a two-sided process or $\mathbb{T} = \mathcal{Z}_+$ for a one-sided process, then the induced probability measure is defined to be the distribution of the process. Thus, for example, a probability space $(\Omega, \mathcal{B}, P)$ together with a doubly infinite sequence of random variables $\{X_n\}_{n\in\mathcal{Z}}$ induces a new probability space $(A^{\mathcal{Z}}, \mathcal{B}_A{}^{\mathcal{Z}}, P_{X^{\mathcal{Z}}})$ and $P_{X^{\mathcal{Z}}}$ is the distribution of the process. For simplicity, let us now denote the process distribution simply by m. We shall call the probability space $(A^{\mathbb{T}}, \mathcal{B}_A{}^{\mathbb{T}}, m)$ induced in this way by a random process $\{X_n\}_{n\in\mathcal{Z}}$ the output space or sequence space of the random process.

Since the sequence space $(A^{\mathbb{T}}, \mathcal{B}_A{}^{\mathbb{T}}, m)$ of a random process $\{X_n\}_{n\in\mathcal{Z}}$ is a probability space, we can define random variables and hence also random processes on this space. One simple and useful such definition is that of a sampling or coordinate or projection function defined as follows: Given a product space $A^{\mathbb{T}}$, define the sampling functions $\Pi_n : A^{\mathbb{T}} \to A$ by

$$\Pi_n(x^{\mathbb{T}}) = x_n, x^{\mathbb{T}} \in A^{\mathbb{T}};\ n \in \mathbb{T}. \tag{1.18}$$

The sampling function is named Π since it is also a projection. Observe that the distribution of the random process $\{\Pi_n\}_{n\in\mathbb{T}}$ defined on the probability space $(A^{\mathbb{T}}, \mathcal{B}_A{}^{\mathbb{T}}, m)$ is exactly the same as the distribution of the random process $\{X_n\}_{n\in\mathbb{T}}$ defined on the probability space $(\Omega, \mathcal{B}, P)$. In fact, so far they are the same process since the $\{\Pi_n\}$ simply read off the values of the $\{X_n\}$.

What happens, however, if we no longer build the Π_n on the X_n, that is, we no longer first select ω from Ω according to P, then form the sequence $x^{\mathbb{T}} = X^{\mathbb{T}}(\omega) = \{X_n(\omega)\}_{n\in\mathbb{T}}$, and then define $\Pi_n(x^{\mathbb{T}}) = X_n(\omega)$? Instead we directly choose an x in $A^{\mathbb{T}}$ using the probability measure m and then view the sequence of coordinate values. In other words, we are considering two completely separate experiments, one described by the probability space $(\Omega, \mathcal{B}, P)$ and the random variables $\{X_n\}$ and the other described by the probability space $(A^{\mathbb{T}}, \mathcal{B}_A{}^{\mathbb{T}}, m)$ and the random variables $\{\Pi_n\}$. In these two separate experiments, the actual sequences selected may be completely different. Yet intuitively the processes should be the "same" in the sense that their statistical structures are identical, that is, they have the same distribution. We make this intuition formal by defining two processes to be *equivalent* if their process distributions are identical, that is, if the probability measures on the output sequence spaces are the same, regardless of the functional form of the random variables of the underlying probability spaces. In the same way, we con-

sider two random variables to be equivalent if their distributions are identical.

We have described above two equivalent processes or two equivalent models for the same random process, one defined as a sequence of random variables on a perhaps very complicated underlying probability space, the other defined as a probability measure directly on the measurable space of possible output sequences. The second model will be referred to as a *directly given* random process or a the *Kolmogorov* model for the random process.

Which model is "better" depends on the application. For example, a directly given model for a random process may focus on the random process itself and not its origin and hence may be simpler to deal with. If the random process is then coded or measurements are taken on the random process, then it may be better to model the encoded random process in terms of random variables defined on the original random process and not as a directly given random process. This model will then focus on the input process and the coding operation. We shall let convenience determine the most appropriate model.

We can now describe yet another model for the above random process, that is, another means of describing a random process with the same distribution. This time the model is in terms of a dynamical system. Given the probability space $(A^{\mathbb{T}}, \mathcal{B}_A{}^{\mathbb{T}}, m)$, define the (left) shift transformation $T : A^{\mathbb{T}} \to A^{\mathbb{T}}$ by

$$T(x^{\mathbb{T}}) = T(\{x_n\}_{n\in\mathbb{T}}) = y^{\mathbb{T}} = \{y_n\}_{n\in\mathbb{T}},$$

where

$$y_n = x_{n+1}, n \in \mathbb{T}.$$

Thus the nth coordinate of $y^{\mathbb{T}}$ is simply the $(n + 1)$st coordinate of $x^{\mathbb{T}}$. (We assume that $\mathbb{T}$ is closed under addition and hence if n and 1 are in $\mathbb{T}$, then so is $(n + 1)$.) If the alphabet of such a shift is not clear from context, we will occasionally denote the shift by T_A or $T_{A^{\mathbb{T}}}$. The shift can easily be shown to be measurable.

Consider next the dynamical system $(A^{\mathbb{T}}, \mathcal{B}_A{}^{\mathbb{T}}, P, T)$ and the random process formed by combining the dynamical system with the zero time sampling function Π_0 (we assume that 0 is a member of $\mathbb{T}$). If we define $Y_n(x) = \Pi_0(T^n x)$ for $x = x^{\mathbb{T}} \in A^{\mathbb{T}}$, or, in abbreviated form, $Y_n = \Pi_0 T^n$, then the random process $\{Y_n\}_{n\in\mathbb{T}}$ is equivalent to the processes developed above. Thus we have developed three different, but equivalent, means of producing the same random process. Each will be seen to have its uses.

The above development shows that a dynamical system is a more fundamental entity than a random process since we can always construct an equivalent model for a random process in terms of a dynamical system — use the directly given representation, shift transformation, and zero

time sampling function. Two important properties of dynamical systems or random processes can be defined at this point, the implications will be developed throughout the book. A dynamical system $(A^{\mathbb{T}}, \mathcal{B}_A{}^{\mathbb{T}}, P, T)$ is said to be *stationary* (with respect to T) if the distribution P is invariant with respect to P, that is,

$$P(T^{-1}F) = P(F), \text{ all } F \in \mathcal{B}_A{}^{\mathbb{T}}. \tag{1.19}$$

In other words, probabilities of process events are unchanged by shifting. The dynamical system is said to be *ergodic* if

$$\text{If } T^{-1}F = F, \text{ then } P(F) = 0 \text{ or } 1, \tag{1.20}$$

that is, all invariant events are trivial. Note that neither definition implies or excludes the other.

The shift transformation on a sequence space introduced above is the most important transformation that we shall encounter. It is not, however, the only important transformation. When dealing with transformations we will usually use the notation T to reflect the fact that it is often related to the action of a simple left shift of a sequence, yet it should be kept in mind that occasionally other operators will be considered and the theory to be developed will remain valid, even if T is not required to be a simple time shift. For example, we will also consider block shifts.

Most texts on ergodic theory deal with the case of an invertible transformation, that is, where T is a one-to-one transformation and the inverse mapping T^{-1} is measurable. This is the case for the shift on $A^{\mathcal{Z}}$, the two-sided shift. It is not the case, however, for the one-sided shift defined on $A^{\mathcal{Z}_+}$ and hence we will avoid use of this assumption. We will, however, often point out in the discussion what simplifications or special properties arise for invertible transformations.

Since random processes are considered equivalent if their distributions are the same, we shall adopt the notation $[A, m, X]$ for a random process $\{X_n; n \in \mathbb{T}\}$ with alphabet A and process distribution m, the index set $\mathbb{T}$ usually being clear from context. We will occasionally abbreviate this to the more common notation $[A, m]$, but it is often convenient to note the name of the output random variables as there may be several, e.g., a random process may have an input X and output Y. By "the associated probability space" of a random process $[A, m, X]$ we shall mean the sequence probability space $(A^{\mathbb{T}}, \mathcal{B}_A{}^{\mathbb{T}}, m)$. It will often be convenient to consider the random process as a directly given random process, that is, to view X_n as the coordinate functions Π_n on the sequence space $A^{\mathbb{T}}$ rather than as being defined on some other abstract space. This will not always be the case, however, as often processes will be formed by coding or communicating other random processes. Context should render such bookkeeping details clear.

1.4 Standard Alphabets

A measurable space $(A, \mathcal{B}_A)$ is a *standard space* if there exists a sequence of finite fields $\mathcal{F}_n$; $n = 1, 2, \cdots$ with the following properties:

(1)$\mathcal{F}_n \subset \mathcal{F}_{n+1}$ (the fields are increasing).
(2)$\mathcal{B}_A$ is the smallest σ-field containing all of the $\mathcal{F}_n$ (the $\mathcal{F}_n$ *generate* $\mathcal{B}_A$ or $\mathcal{B}_A = \sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$).
(3)An event $G_n \in \mathcal{F}_n$ is called an *atom* of the field if it is nonempty and and its only subsets which are also field members are itself and the empty set. If $G_n \in \mathcal{F}_n$; $n = 1, 2, \cdots$ are atoms and $G_{n+1} \subset G_n$ for all n, then

$$\bigcap_{n=1}^{\infty} G_n \neq \varnothing.$$

Standard spaces are important for several reasons: First, they are a general class of spaces for which two of the key results of probability hold: (1) the Kolmogorov extension theorem showing that a random process is completely described by its finite order distributions, and (2) the existence of regular conditional probability measures. Thus, in particular, the conditional probability measure $P_{X|Y}(F|y)$ of (1.13) is regular if the alphabets A_X and A_Y are standard and hence for each fixed $y \in A_Y$ the set function $P_{X|Y}(F|y)$; $F \in \mathcal{B}_{A_X}$ is a probability measure. In this case we can interpret $P_{X|Y}(F|y)$ as $P(X \in F|Y = y)$. Second, the ergodic decomposition theorem of ergodic theory holds for such spaces. The ergodic decomposition implies that any stationary process is equivalent to a mixture of stationary and ergodic processes; that is, a stationary nonergodic source can be viewed as a random selection of one of a family of stationary and ergodic sources. Third, the class is sufficiently general to include virtually all examples arising in applications, e.g., discrete spaces, the real line, Euclidean vector spaces, Polish spaces (complete separable metric spaces), etc. The reader is referred to [55] or [58] and the references cited therein for a detailed development of these properties and examples of standard spaces.

Standard spaces are not the most general space for which the Kolmogorov extension theorem, the existence of conditional probability, and the ergodic decomposition theorem all hold. These results also hold for perfect spaces which include standard spaces as a special case. (See, e.g., [161],[174],[155], [114].) We limit discussion to standard spaces, however, as they are easier to characterize and work with and they are sufficiently general to handle most cases encountered in applications. Although standard spaces are not the most general for which the required probability theory results hold, they are the most general for which all finitely additive normalized measures extend to countably additive prob-

ability measures, a property which greatly eases the proof of many of the desired results.

Throughout this book we shall assume that the alphabet A of the information source is a standard space.

1.5 Expectation

Let $(\Omega, \mathcal{B}, m)$ be a probability space, e.g., the probability space of a directly given random process with alphabet A, $(A^{\mathbb{T}}, B_A{}^{\mathbb{T}}, m)$. A real-valued random variable $f : \Omega \to \mathbb{R}$ will also be called a *measurement* since it is often formed by taking a mapping or function of some other set of more general random variables, e.g., the outputs of some random process which might not have real-valued outputs. Measurements made on such processes, however, will always be assumed to be real.

Suppose next we have a measurement f whose range space or *alphabet* $f(\Omega) \subset \mathbb{R}$ of possible values is finite. Then f is called a *discrete random variable* or *discrete measurement* or *digital measurement* or, in the common mathematical terminology, a *simple function*.

Given a discrete measurement f, suppose that its range space is $f(\Omega) = \{b_i, i = 1, \cdots, N\}$, where the b_i are distinct. Define the sets $F_i = f^{-1}(b_i) = \{x : f(x) = b_i\}$, $i = 1, \cdots, N$. Since f is measurable, the F_i are all members of $\mathcal{B}$. Since the b_i are distinct, the F_i are disjoint. Since every input point in Ω must map into some b_i, the union of the F_i equals Ω. Thus the collection $\{F_i; i = 1, 2, \cdots, N\}$ forms a partition of Ω. We have therefore shown that any discrete measurement f can be expressed in the form

$$f(x) = \sum_{i=1}^{M} b_i 1_{F_i}(x), \tag{1.21}$$

where $b_i \in \mathbb{R}$, the $F_i \in \mathcal{B}$ form a partition of Ω, and 1_{F_i} is the indicator function of F_i, $i = 1, \cdots, M$. Every simple function has a unique representation in this form with distinct b_i and $\{F_i\}$ a partition.

The *expectation* or *ensemble average* or *probabilistic average* or *mean* of a discrete measurement $f : \Omega \to \mathbb{R}$ as in (1.21) with respect to a probability measure m is defined by

$$E_m f = \sum_{i=0}^{M} b_i m(F_i). \tag{1.22}$$

An immediate consequence of the definition of expectation is the simple but useful fact that for any event F in the original probability space,

$$E_m 1_F = m(F),$$

that is, probabilities can be found from expectations of indicator functions.

Again let $(\Omega, \mathcal{B}, m)$ be a probability space and $f : \Omega \to \mathbb{R}$ a measurement, that is, a real-valued random variable or measurable real-valued function. Define the sequence of *quantizers* $q_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \cdots$, as follows:

$$q_n(r) = \begin{cases} n & n \le r \\ (k-1)2^{-n} & (k-1)2^{-n} \le r < k2^{-n},\ k = 1, 2, \cdots, n2^n \\ -(k-1)2^{-n} & -k2^{-n} \le r < -(k-1)2^{-n};\ k = 1, 2, \cdots, n2^n \\ -n & r < -n. \end{cases}$$

We now define expectation for general measurements in two steps. If $f \ge 0$, then define

$$E_m f = \lim_{n \to \infty} E_m(q_n(f)). \tag{1.23}$$

Since the q_n are discrete measurements on f, the $q_n(f)$ are discrete measurements on Ω ($q_n(f)(x) = q_n(f(x))$ is a simple function) and hence the individual expectations are well defined. Since the $q_n(f)$ are nondecreasing, so are the $E_m(q_n(f))$ and this sequence must either converge to a finite limit or grow without bound, in which case we say it converges to ∞. In both cases the expectation $E_m f$ is well defined, although it may be infinite.

If f is an arbitrary real random variable, define its positive and negative parts $f^+(x) = \max(f(x), 0)$ and $f^-(x) = -\min(f(x), 0)$ so that $f(x) = f^+(x) - f^-(x)$ and set

$$E_m f = E_m f^+ - E_m f^- \tag{1.24}$$

provided this does not have the form $+\infty - \infty$, in which case the expectation does not exist. It can be shown that the expectation can also be evaluated for nonnegative measurements by the formula

$$E_m f = \sup_{\text{discrete } g:\, g \le f} E_m g.$$

The expectation is also called an *integral* and is denoted by any of the following:

$$E_m f = \int f dm = \int f(x) dm(x) = \int f(x) m(dx).$$

The subscript m denoting the measure with respect to which the expectation is taken will occasionally be omitted if it is clear from context.

A measurement f is said to be *integrable* or *m-integrable* if $E_m f$ exists and is finite. A function is integrable if and only if its absolute value is

integrable. Define $L^1(m)$ to be the space of all m-integrable functions. Given any m-integrable f and an event B, define

$$\int_B f\,dm = \int f(x)1_B(x)\,dm(x).$$

Two random variables f and g are said to be equal m-almost-everywhere or equal m-a.e. or equal with m-probability one if $m(f = g) = m(\{x : f(x) = g(x)\}) = 1$. The m- is dropped if it is clear from context.

Given a probability space $(\Omega, \mathcal{B}, m)$, suppose that $\mathcal{G}$ is a sub-σ-field of $\mathcal{B}$, that is, it is a σ-field of subsets of Ω and all those subsets are in $\mathcal{B}$ ($\mathcal{G} \subset \mathcal{B}$). Let $f : \Omega \to \mathbb{R}$ be an integrable measurement. Then the *conditional expectation* $E(f|\mathcal{G})$ is described as any function, say $h(\omega)$, that satisfies the following two properties:

$$h(\omega) \text{ is measurable with respect to } \mathcal{G} \tag{1.25}$$

$$\int_G h\,dm = \int_G f\,dm; \text{ all } G \in \mathcal{G}. \tag{1.26}$$

If a regular conditional probability distribution given $\mathcal{G}$ exists, e.g., if the space is standard, then one has a constructive definition of conditional expectation: $E(f|\mathcal{G})(\omega)$ is simply the expectation of f with respect to the conditional probability measure $m(.|\mathcal{G})(\omega)$. Applying this to the example of two random variables X and Y with standard alphabets described in Section 1.2 we have from (1.26) that for integrable $f : A_X \times A_Y \to \mathbb{R}$

$$E(f) = \int f(x,y)dP_{XY}(x,y) = \int (\int f(x,y)dP_{X|Y}(x|y))dP_Y(y). \tag{1.27}$$

In particular, for fixed y, $f(x,y)$ is an integrable (and measurable) function of x.

Equation (1.27) provides a generalization of (1.13) from rectangles to arbitrary events. For an arbitrary $F \in \mathcal{B}_{A_X \times A_Y}$ we have that

$$P_{XY}(F) = \int \left(\int 1_F(x,y)dP_{X|Y}(x|y) \right) dP_Y(y) = \int P_{X|Y}(F_y|y)dP_Y(y), \tag{1.28}$$

where $F_y = \{x : (x,y) \in F\}$ is called the *section* of F at y. If F is measurable, then so is F_y for all y. Alternatively, since $1_F(x,y)$ is measurable with respect to x for each fixed y, $F_y \in \mathcal{B}_{A_X}$ and the inner integral is just

$$\int_{x:(x,y)\in F} dP_{X|Y}(x|y) = P_{X|Y}(F_y|y).$$

1.6 Asymptotic Mean Stationarity

Recall that a dynamical system (or the associated source) $(\Omega, \mathcal{B}, P, T)$ is said to be stationary if $P(T^{-1}G) = P(G)$ for all $G \in \mathcal{B}$. It is said to be *asymptotically mean stationary* or, simply, AMS if the limit

$$\overline{P}(G) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P(T^{-k}G) \tag{1.29}$$

exists for all $G \in \mathcal{B}$. The following theorems summarize several important properties of AMS sources. Details may be found in Chapter 6 of [55] or Chapter 7 of [58].

Theorem 1.2. *If a dynamical system $(\Omega, \mathcal{B}, P, T)$ is AMS, then $\overline{P}$ defined in (1.29) is a probability measure and $(\Omega, \mathcal{B}, \overline{P}, T)$ is stationary. The distribution $\overline{P}$ is called the stationary mean of P. If an event G is invariant in the sense that $T^{-1}G = G$, then*

$$P(G) = \overline{P}(G).$$

If a random variable g is invariant in the sense that $g(Tx) = g(x)$ with P probability 1, then

$$E_P g = E_{\overline{P}} g.$$

The stationary mean $\overline{P}$ *asymptotically dominates* P in the sense that if $\overline{P}(G) = 0$, then

$$\limsup_{n\to\infty} P(T^{-n}G) = 0.$$

Theorem 1.3. *Given an AMS source $\{X_n\}$ let $\sigma(X_n, X_{n+1}, \cdots)$ denote the σ-field generated by the random variables $X_n, \cdots$, that is, the smallest σ-field with respect to which all these random variables are measurable. Define the tail σ-field $\mathcal{F}_\infty$ by*

$$\mathcal{F}_\infty = \bigcap_{n=0}^{\infty} \sigma(X_n, \cdots).$$

If $G \in \mathcal{F}_\infty$ and $\overline{P}(G) = 0$, then also $P(G) = 0$.

The tail σ-field can be thought of as events that are determinable by looking only at samples of the sequence in the arbitrarily distant future. The theorem states that the stationary mean *dominates* the original measure on such tail events in the sense that zero probability under the stationary mean implies zero probability under the original source.

1.7 Ergodic Properties

Two of the basic results of ergodic theory that will be called upon extensively are the pointwise or almost-everywhere ergodic theorem and the ergodic decomposition theorem We quote these results along with some relevant notation for reference. Detailed developments may be found in Chapters 6–8 of [55] or Chapters 7–10 of [58]. The ergodic theorem states that AMS dynamical systems (and hence also sources) have convergent sample averages, and it characterizes the limits.

Theorem 1.4. *If a dynamical system $(\Omega, \mathcal{B}, m, T)$ is AMS with stationary mean $\overline{m}$ and if $f \in L^1(\overline{m})$, then with probability one under m and $\overline{m}$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} fT^i = E_{\overline{m}}(f|\mathcal{I}),$$

where $\mathcal{I}$ is the sub-σ-field of invariant events, that is, events G for which $T^{-1}G = G$.

The basic idea of the ergodic decomposition is that any stationary source which is not ergodic can be represented as a mixture of stationary ergodic components or subsources.

Theorem 1.5. Ergodic Decomposition *Given the standard sequence space $(\Omega, \mathcal{B})$ with shift T as previously, there exists a family of stationary ergodic measures $\{p_x;\ x \in \Omega\}$, called the ergodic decomposition, with the following properties:*

(a) $p_{Tx} = p_x$.
(b)For any stationary measure m,

$$m(G) = \int p_x(G)\, dm(x);\ \text{all } G \in \mathcal{B}.$$

(c)For any $g \in L^1(m)$

$$\int g\, dm = \int \left(\int g\, dp_x \right) dm(x).$$

It is important to note that the same collection of stationary ergodic components works for any stationary measure m. This is the strong form of the ergodic decomposition.

The final result of this section is a variation on the ergodic decomposition. To describe the result, we need to digress briefly to introduce a metric on spaces of probability measures. A thorough development can be found in Chapter 8 of [55] or Chapter 9 of [58]. We have a standard sequence measurable space $(\Omega, \mathcal{B})$ and hence we can generate the σ-field $\mathcal{B}$

by a countable field $\mathcal{F} = \{F_n; n = 1, 2, \cdots\}$. Given such a countable generating field, a *distributional distance* between two probability measures p and m on $(\Omega, \mathcal{B})$ is defined by

$$d(p, m) = \sum_{n=1}^{\infty} 2^{-n} |p(F_n) - m(F_n)|.$$

Any choice of a countable generating field yields a distributional distance. Such a distance or metric yields a measurable space of probability measures as follows: Let Λ denote the space of all probability measures on the original measurable space $(\Omega, \mathcal{B})$. Let $\mathcal{B}(\Lambda)$ denote the σ-field of subsets of Λ generated by all open spheres using the distributional distance, that is, all sets of the form $\{p : d(p, m) \leq \epsilon\}$ for some $m \in \Lambda$ and some $\epsilon > 0$. We can now consider properties of functions that carry sequences in our original space into probability measures. The following is Theorem 8.5.1 of [55] and Theorem 10.1 of [58].

Theorem 1.6. A Variation on the Ergodic Decomposition *Fix a standard measurable space $(\Omega, \mathcal{B})$ and a transformation $T : \Omega \to \Omega$. Then there are a standard measurable space $(\Lambda, \mathcal{L})$, a family of stationary ergodic measures $\{m_\lambda; \lambda \in \Lambda\}$ on $(\Omega, \mathcal{B})$, and a measurable mapping $\psi : \Omega \to \Lambda$ such that*

(a)ψ is invariant ($\psi(Tx) = \psi(x)$ all x);
(b)if m is a stationary measure on $(\Omega, \mathcal{B})$ and P_ψ is the induced distribution; that is, $P_\psi(G) = m(\psi^{-1}(G))$ for $G \in \Lambda$ (which is well defined from (a)), then

$$m(F) = \int dm(x) m_{\psi(x)}(F) = \int dP_\psi(\lambda) m_\lambda(F), \text{ all } F \in \mathcal{B},$$

and if $f \in L^1(m)$, then so is $\int f dm_\lambda$ P_ψ-a.e. and

$$E_m f = \int dm(x) E_{m_{\psi(x)}} f = \int dP_\psi(\lambda) E_{m_\lambda} f.$$

Finally, for any event F, $m_\psi(F) = m(F|\psi)$, that is, given the ergodic decomposition and a stationary measure m , the ergodic component λ is a version of the conditional probability under m given $\psi = \lambda$.

The following corollary to the ergodic decomposition is Lemma 8.6.2 of [55] and Lemma 10.4 of [58]. It states that the conditional probability of a future event given the entire past is unchanged by knowing the ergodic component in effect. This is because the infinite past determines the ergodic component in effect.

Corollary 1.1. *Suppose that $\{X_n\}$ is a two-sided stationary process with distribution m and that $\{m_\lambda; \lambda \in \Lambda\}$ is the ergodic decomposition and ψ*

the ergodic component function. Then the mapping ψ is measurable with respect to $\sigma(X_{-1}, X_{-2}, \cdots)$ and

$$m((X_0, X_1, \cdots) \in F | X_{-1}, X_{-2}, \cdots) = m_\psi((X_0, X_1, \cdots) \in F | X_{-1}, X_{-2}, \cdots);\ m - \text{a.e.}$$

Chapter 2
Pair Processes: Channels, Codes, and Couplings

Abstract We have considered a random process or source $\{X_n\}$ as a sequence of random entities, where the object produced at each time could be quite general, e.g., a random variable, vector, or waveform. Hence sequences of pairs of random objects such as $\{X_n, Y_n\}$ are included in the general framework. We now focus on the possible interrelations between the two components of such a pair process. First consider the situation where we begin with one source, say $\{X_n\}$, called the *input* and use either a random or a deterministic mapping of the input sequence $\{X_n\}$ to form an output sequence $\{Y_n\}$. We generally refer to the mapping as a *channel* if it is random and a *code* if it is deterministic. Hence a code is a special case of a channel and results for channels will immediately imply corresponding results for codes. The initial point of interest will be conditions on the structure of the channel under which the resulting pair process $\{X_n, Y_n\}$ will inherit stationarity and ergodic properties from the original source $\{X_n\}$. We will also be interested in the behavior resulting when the output of one channel serves as the input to another, that is, when we form a new channel as a cascade of other channels. Such cascades yield models of a communication system which typically has a code mapping (called the *encoder*) followed by a channel followed by another code mapping (called the *decoder*). Lastly, pair processes arise naturally in other situations, including *coupling* two separate processes by constructing a joint distribution. This chapter develops the context for the development in future chapters of the properties of information and entropy arising in pair processes.

2.1 Pair Processes

A common object throughout this book and the focus of this chapter is the idea of a pair process. The notation will vary somewhat depend-

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ing on the specific application, but basically a pair process is a random process with two components, e.g., a sequence of random variables $\{(X_n, Y_n); n \in \mathbb{T}\}$ with alphabets A and B and process distribution p on $(A^{\mathbb{T}} \times B^{\mathbb{T}}, \mathcal{B}(A^{\mathbb{T}} \times B^{\mathbb{T}}))$. When we wish to emphasize the names of the separate component random variables and processes, we will often write A_X for A and A_Y for B and p_{XY} for p. The process X or $\{X_n\}$ will often have the interpretation of being the input of a code or channel or a cascade of such operations and Y or $\{Y_n\}$ the output. A pair process induces two "marginal" process $\{X_n\}$ with process distribution, say μ, and $\{Y_n\}$, with process distribution η. When we wish to emphasize the random variables we might write p_X or μ_X instead of μ and p_Y or μ_Y or η_Y instead of η. All of these notations have their uses, and the added subscripts often help sort out which random process or variables are important. Often we will use $\hat{X}_n$ as the second component instead of Y when it is viewed as an approximation to the first component X_n.

2.2 Channels

A channel converts one information source – typically called the *input* to the channel – into another – called the *output*. In general the operation is random and is specified by a conditional probability measure of output sequences given an input sequence. The combination of an input distribution with the channel yields a pair process, a process with an input component and an output component. If the channel is deterministic rather than random, the operation is called a *code*. In this section the basic definitions of channels and codes are introduced.

A fundamental nuisance in the development of channels and codes is the notion of time. So far we have considered pair processes where at each unit of time, one random object is produced for each coordinate of the pair. In the channel or code example, this corresponds to one output for every input. Interesting communication systems do not always easily fit into this framework, and this can cause serious problems in notation and in the interpretation and development of results. For example, suppose that an input source consists of a sequence of real numbers and let T denote the time shift on the real sequence space. Suppose that the output source consists of a binary sequence and let S denote its shift. Suppose also that the channel is such that for each real number in, three binary symbols are produced. This fits our usual framework if we consider each output variable to consist of a binary three-tuple since then there is one output vector for each input symbol. One must be careful, however, when considering the stationarity of such a system. Do we consider the output process to be physically stationary if it is stationary with respect to S or with respect to S^3? The former might make more

sense if we are looking at the output alone, the latter if we are looking at the output in relation to the input. How do we define stationarity for the pair process? Given two sequence spaces, we might first construct a shift on the pair sequence space as simply the cartesian product of the shifts, e.g., given an input sequence x and an output sequence y define a shift T^* by $T^*(x,y) = (Tx, Sy)$. While this might seem natural given only the pair random process $\{X_n, Y_n\}$, it is not natural in the physical context that one symbol of X yields three symbols of Y. In other words, the two shifts do not correspond to the same amount of *time*. Here the more physically meaningful shift on the pair space would be $T'(x,y) = (Tx, S^3y)$ and the more physically meaningful questions on stationarity and ergodicity relate to T' and not to T^*. The problem becomes even more complicated when channels or codes produce a varying number of output symbols for each input symbol, where the number of symbols depends on the input sequence. Such variable rate codes arise often in practice, especially for noiseless coding applications such as Huffman, Lempel-Ziv, and arithmetic codes. While we will not treat such variable rate systems in any detail, they point out the difficulty that can arise associating the mathematical shift operation with physical time when we are considering cartesian products of spaces, each having their own shift.

There is no easy way to solve this problem notationally. We adopt the following view as a compromise which is usually adequate for fixed-rate systems. We will be most interested in pair processes that are stationary in the physical sense, that is, whose statistics are not changed when both are shifted by an equal amount of *physical* time. This is the same as stationarity with respect to the product shift if the two shifts correspond to equal amounts of physical time. Hence for simplicity we will usually focus on this case. More general cases will be introduced when appropriate to point out their form and how they can be put into the matching shift structure by considering groups of symbols and different shifts. This will necessitate occasional discussions about what is meant by stationarity or ergodicity for a particular system.

The mathematical generalization of Shannon's original notions of sources, codes, and channels are due to Khinchine [87] [88]. Khinchine's results characterizing stationarity and ergodicity of channels were corrected and developed by Adler [2].

Say we are given a source $[A, X, \mu]$, that is, a sequence of A-valued random variables $\{X_n; n \in \mathbb{T}\}$ defined on a common probability space $(\Omega, \mathcal{F}, P)$ having a process distribution μ defined on the measurable sequence space $(B^{\mathbb{T}}, \mathcal{B}_A{}^{\mathbb{T}})$. We shall let $X = \{X_n; n \in \mathbb{T}\}$ denote the sequence-valued random variable, that is, the random variable taking values in $A^{\mathbb{T}}$ according to the distribution μ. Let B be another alphabet with a corresponding measurable sequence space $(A^{\mathbb{T}}, \mathcal{B}_B{}^{\mathbb{T}})$. We assume as usual that A and B are standard and hence so are their sequence

spaces and cartesian products. A *channel* $[A, \nu, B]$ with input alphabet A and output alphabet B (we denote the channel simply by ν when these alphabets are clear from context) is a family of probability measures $\{\nu_x; x \in A^{\mathbb{T}}\}$ on $(B^{\mathbb{T}}, \mathcal{B}_B{}^{\mathbb{T}})$ (the output sequence space) such that for every output event $F \in \mathcal{B}_B{}^{\mathbb{T}}$ $\nu_x(F)$ is a measurable function of x. This measurability requirement ensures that the set function p specified on the joint input/output space $(A^{\mathbb{T}} \times B^{\mathbb{T}})$, $\mathcal{B}_A{}^{\mathbb{T}} \times \mathcal{B}_B{}^{\mathbb{T}})$ by its values on rectangles as

$$p(G \times F) = \int_G d\mu(x)\nu_x(F); F \in \mathcal{B}_B{}^{\mathbb{T}}, G \in \mathcal{B}_A{}^{\mathbb{T}},$$

is well defined. The set function p is nonnegative, normalized, and countably additive on the field generated by the rectangles $G \times F$, $G \in \mathcal{B}_A{}^{\mathbb{T}}$, $F \in \mathcal{B}_B{}^{\mathbb{T}}$. Thus p extends to a probability measure on the joint input/output space, which is sometimes called the *hookup* of the source μ and channel ν. We will often denote this joint measure by $\mu\nu$. The corresponding sequences of random variables are called the *input/output process*.

Thus a channel is a probability measure on the output sequence space for each input sequence such that a joint input/output probability measure is well-defined. The above equation shows that a channel is simply a regular conditional probability, in particular,

$$\nu_x(F) = p((x, y) : y \in F|x); F \in \mathcal{B}_B{}^{\mathbb{T}}, x \in A^{\mathbb{T}}.$$

We can relate a channel to the notation used previously for conditional distributions by using the sequence-valued random variables $X = \{X_n; n \in \mathbb{T}\}$ and $Y = \{Y_n; n \in \mathbb{T}\}$:

$$\nu_x(F) = P_{Y|X}(F|x). \tag{2.1}$$

Eq. (1.28) then provides the probability of an arbitrary input/output event:

$$p(F) = \int d\mu(x)\nu_x(F_x),$$

where $F_x = \{y : (x, y) \in F\}$ is the *section* of F at x.

If we start with a hookup p, then we can obtain the input distribution μ as

$$\mu(F) = p(F \times B^{\mathbb{T}}); F \in \mathcal{B}_A{}^{\mathbb{T}}.$$

Similarly we can obtain the output distribution, say η, via

$$\eta(F) = p(A^{\mathbb{T}} \times F); F \in \mathcal{B}_B{}^{\mathbb{T}}.$$

Suppose one now starts with a pair process distribution p and hence also with the induced source distribution μ. Does there exist a channel ν

for which $p = \mu\nu$? The answer is yes since the spaces are standard. One can always define the conditional probability $\nu_x(F) = P(F \times A^{\mathbb{T}}|X = x)$ for all input sequences x, but this need not possess a regular version, that is, be a probability measure for all x, in the case of arbitrary alphabets. If the alphabets are standard, however, we have seen that a regular conditional probability measure always exists.

2.3 Stationarity Properties of Channels

We now define a variety of stationarity properties for channels that are related to, but not the same as, those for sources. The motivation behind the various definitions is that stationarity properties of channels coupled with those of sources should imply stationarity properties for the resulting source-channel hookups.

The classical definition of a stationary channel is the following: Suppose that we have a channel $[A, \nu, B]$ and suppose that T_A and T_B are the shifts on the input sequence space and output sequence space, respectively. The channel is *stationary* with respect to T_A and T_B or (T_A, T_B)-*stationary* if

$$\nu_x(T_B^{-1}F) = \nu_{T_Ax}(F), x \in \mathcal{A}^{\mathbb{T}}, F \in \mathcal{B}_B{}^{\mathbb{T}}. \tag{2.2}$$

If the transformations are clear from context then we simply say that the channel is stationary. Intuitively, a right shift of an output event yields the same probability as the left shift of an input event. The different shifts are required because in general only T_Ax and not $T_A^{-1}x$ exists since the shift may not be invertible and in general only $T_B^{-1}F$ and not T_BF exists for the same reason. If the shifts are invertible, e.g., the processes are two-sided, then the definition is equivalent to

$$\nu_{T_Ax}(T_BF) = \nu_{T_A^{-1}x}(T_B^{-1}F) = \nu_x(F), \text{ all } x \in A^{\mathbb{T}}, F \in \mathcal{B}_B{}^{\mathbb{T}} \tag{2.3}$$

that is, shifting the input sequence and output event in the same direction does not change the probability.

The fundamental importance of the stationarity of a channel is contained in the following lemma.

Lemma 2.1. *If a source $[A, \mu]$, stationary with respect to T_A, is connected to channel $[A, \nu, B]$, stationary with respect to T_A and T_B, then the resulting hookup $\mu\nu$ is stationary with respect to the cartesian product shift $T = T_{A\times B} = T_A \times T_B$ defined by $T(x, y) = (T_Ax, T_By)$.*

Proof: We have that

$$\mu\nu(T^{-1}F) = \int d\mu(x)\nu_x((T^{-1}F)_x).$$

Now

$$(T^{-1}F)_x = \{y : T(x,y) \in F\} = \{y : (T_A x, T_B y) \in F\}$$
$$= \{y : T_B y \in F_{T_A x}\} = T_B^{-1} F_{T_A x}$$

and hence

$$\mu\nu(T^{-1}F) = \int d\mu(x)\nu_x(T_B^{-1}F_{T_A x}).$$

Since the channel is stationary, however, this becomes

$$\mu\nu(T^{-1}F) = \int d\mu(x)\nu_{T_A x}(F_{T_A x}) = \int d\mu T_A^{-1}(x)\nu_x(F_x),$$

where we have used the change of variables formula. Since μ is stationary, however, the right hand side is

$$\int d\mu(x)\nu_x(F),$$

which proves the lemma. □

Suppose next that we are told that a hookup $\mu\nu$ is stationary. Does it then follow that the source μ and channel ν are necessarily stationary? The source must be since

$$\mu(T_A^{-1}F) = \mu\nu((T_A \times T_B)^{-1}(F \times B^{\mathbb{T}})) = \mu\nu(F \times B^{\mathbb{T}}) = \mu(F).$$

The channel need not be stationary, however, since, for example, the stationarity could be violated on a set of μ measure 0 without affecting the proof of the above lemma. This suggests a somewhat weaker notion of stationarity which is more directly related to the stationarity of the hookup. We say that a channel $[A, \nu, B]$ is *stationary with respect to a source* $[A, \mu]$ if $\mu\nu$ is stationary. We also state that a channel is stationary μ-a.e. if it satisfies (2.2) for all x in a set of μ-probability one. If a channel is stationary μ-a.e. and μ is stationary, then the channel is also stationary with respect to μ. Clearly a stationary channel is stationary with respect to all stationary sources. The reason for this more general view is that we wish to extend the definition of stationary channels to asymptotically mean stationary channels. The general definition extends; the classical definition of stationary channels does not.

Observe that the various definitions of stationarity of channels immediately extend to block shifts since they hold for any shifts defined on the input and output sequence spaces, e.g., a channel stationary with respect to T_A^N and T_B^K could be a reasonable model for a channel or code that puts out K symbols from an alphabet B every time it takes in N symbols from an alphabet A. We shorten the name (T_A^N, T_B^K)-stationary

to (N,K)-stationary channel in this case. A stationary channel (without modifiers) is simply a (1,1)-stationary channel in this sense.

The most general notion of stationarity that we are interested in is that of asymptotic mean stationarity We define a channel $[A,\nu,B]$ to be *asymptotically mean stationary* or *AMS* for a source $[A,\mu]$ with respect to T_A and T_B if the hookup $\mu\nu$ is AMS with respect to the product shift $T_A \times T_B$. As in the stationary case, an immediate necessary condition is that the input source be AMS with respect to T_A. A channel will be said to be (T_A,T_B)-AMS if the hookup is (T_A,T_B)-AMS for all T_A-AMS sources.

The following lemma shows that an AMS channel is indeed a generalization of the idea of a stationary channel and that the stationary mean of a hookup of an AMS source to a stationary channel is simply the hookup of the stationary mean of the source to the channel.

Lemma 2.2. *Suppose that ν is (T_A,T_B)-stationary and that μ is AMS with respect to T_A. Let $\overline{\mu}$ denote the stationary mean of μ and observe that $\overline{\mu}\nu$ is stationary. Then the hookup $\mu\nu$ is AMS with stationary mean*

$$\overline{\mu\nu} = \overline{\mu}\nu.$$

Thus, in particular, ν is an AMS channel.

Proof: We have that

$$\begin{aligned}(T^{-i}F)_x &= \{y : (x,y) \in T^{-i}F\} = \{y : T^i(x,y) \in F\}\\ &= \{y : (T_A^i x, T_B^i y) \in F\} = \{y : T_B^i y \in F_{T_A^i x}\}\\ &= T_B^{-i} F_{T_A^i x}\end{aligned}$$

and therefore since ν is stationary

$$\begin{aligned}\mu\nu(T^{-i}F) &= \int d\mu(x)\nu_x(T_B^{-i}F_{T_A^i x})\\ &= \int d\mu(x)\nu_{T_A^i x}(F_{T_A^i x}) = \int d\mu T_A^{-i}(x)\nu_x(F).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1}\mu\nu(T^{-i}F) &= \frac{1}{n}\sum_{i=0}^{n-1}\int d\mu T_A^{-i}(x)\nu_x(F)\\ &\underset{n\to\infty}{\to} \int d\overline{\mu}(x)\nu_x(F) = \overline{\mu}\nu(F)\end{aligned}$$

from Lemma 6.5.1 of [55] or Lemma 7.9 if [58]. This proves that $\mu\nu$ is AMS and that the stationary mean is $\overline{\mu}\nu$. □

A final property crucial to quantifying the behavior of random processes is that of ergodicity. Hence we define a (stationary, AMS) channel

ν to be ergodic with respect to (T_A, T_B) if it has the property that whenever a (stationary, AMS) ergodic source (with respect to T_A) is connected to the channel, the overall input/output process is (stationary, AMS) ergodic. The following modification of Lemma 6.7.4 of [55] or Lemma 7.15 of [58] is the principal tool for proving a channel to be ergodic.

Lemma 2.3. *An AMS (stationary) channel $[A, \nu, B]$ is ergodic if for all AMS (stationary) sources μ and all sets of the form $\overline{F} = F_A \times F_B$, $\overline{G} = G_A \times G_B$ for rectangles $F_A, G_A \in \mathcal{B}_A^\infty$ and $F_B, G_B \in \mathcal{B}_B^\infty$ we have that for $p = \mu\nu$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} p(T_{A\times B}^{-i} \overline{F} \bigcap \overline{G}) = \overline{p}(\overline{F}) p(\overline{G}), \tag{2.4}$$

where $\overline{p}$ is the stationary mean of p (p if p is already stationary).

Proof: The proof parallels that of Lemma 6.7.4 of [55] or Lemma 7.15 of [58]. The result does not follow immediately from that lemma since the collection of given sets does not itself form a field. Arbitrary events $F, G \in \mathcal{B}_{A\times B}^\infty$ can be approximated arbitrarily closely by events in the field generated by the above rectangles and hence given $\epsilon > 0$ we can find finite disjoint rectangles of the given form F_i, G_i, $i = 1, \cdots, L$ such that if $F_0 = \bigcup_{i=1}^L F_i$ and $G_0 = \bigcup_{i=1}^L G_i$, then $p(F\Delta F_0)$, $p(G\Delta G_0)$, $\overline{p}(F\Delta F_0)$, and $\overline{p}(G\Delta G_0)$ are all less than ϵ. Then

$$\begin{aligned}
&|\frac{1}{n} \sum_{k=0}^{n-1} p(T^{-k}F \bigcap G) - \overline{p}(F)p(G)| \le \\
&\qquad |\frac{1}{n} \sum_{k=0}^{n-1} p(T^{-k}F \bigcap G) - \frac{1}{n} \sum_{k=0}^{n-1} p(T^{-k}F_0 \bigcap G_0)| + \\
&|\frac{1}{n} \sum_{k=0}^{n-1} p(T^{-k}F_0 \bigcap G_0) - \overline{p}(F_0)p(G_0)| + |\overline{p}(F_0)p(G_0) - \overline{p}(F)p(G)|.
\end{aligned}$$

Exactly as in Lemma 6.7.4 of [55], the rightmost term is bound above by 2ϵ and the first term on the left goes to zero as $n \to \infty$. The middle term is the absolute magnitude of

$$\begin{aligned}
&\frac{1}{n} \sum_{k=0}^{n-1} p(T^{-k} \bigcup_i F_i \bigcap \bigcup_j G_j) - \overline{p}(\bigcup_i F_i) p(\bigcup_j G_j) = \\
&\qquad \sum_{i,j} \left(\frac{1}{n} \sum_{k=0}^{n-1} p(T^{-k}F_i \bigcap G_j) - \overline{p}(F_i)p(G_j) \right).
\end{aligned}$$

Each term in the finite sum converges to 0 by assumption. Thus p is ergodic from Lemma 6.7.4 of [55] or Lemma 7.15 of [58]. □

Because of the specific class of sets chosen, the above lemma considered separate sets for shifting and remaining fixed, unlike using the same set for both purposes as in Lemma 6.7.4 of [55] or Lemma 7.15 of [58]. This was required so that the cross products in the final sum considered would converge accordingly.

2.4 Extremes: Noiseless and Completely Random Channels

The first two examples of channels are the simplest, the first doing nothing to the input but reproducing it perfectly and the second being useless (at least for communication purposes) since the output is random and independent of the input. Both extremes provide simple examples of the properties of channels, and the completely random example will reappear when applying channel structure ideas to sources.

Noiseless Channels

A channel $[A, \nu, B]$ is said to be *noiseless* if $A = B$ and

$$\nu_x(F) = \begin{cases} 1 & x \in F \\ 0 & x \notin F \end{cases}$$

that is, with probability one the channel puts out what goes in, it acts as an ideal wire. In engineering terms, it is discrete-time linear system with impulse response equal to an impulse.

A noiseless channel is clearly stationary and ergodic.

Completely Random Channels

Suppose that η is a probability measure on the output space $(B^{\mathbb{T}}, \mathcal{B}_B{}^{\mathbb{T}})$ and define a channel

$$\nu_x(F) = \eta(F), F \in \mathcal{B}_B{}^{\mathbb{T}}, x \in A^{\mathbb{T}}.$$

Then it is easy to see that the input/output measure satisfies

$$p(G \times F) = \eta(F)\mu(G); F \in \mathcal{B}_B{}^{\mathbb{T}}, G \in \mathcal{B}_A{}^{\mathbb{T}},$$

and hence the input/output measure is a product measure and the input and output sequences are therefore independent of each other. This

channel is called a *completely random channel* or *product channel* because the output is independent of the input.

This channel is quite useless because the output tells us nothing of the input. The completely random channel is stationary (AMS) if the measure η is stationary (AMS). Perhaps surprisingly, such a channel need not be ergodic even if η is ergodic since the product of two stationary and ergodic sources need not be ergodic. (See, e.g., [22].) We shall later see that if η is also assumed to be weakly mixing, then the resulting channel is ergodic.

A generalization of the noiseless channel that is of much greater interest is the deterministic channel. Here the channel is not random, but the output is formed by a general mapping of the input rather than being the input itself.

2.5 Deterministic Channels and Sequence Coders

A channel $[A, \nu, B]$ is said to be *deterministic* if each input string x is mapped into an output string $f(x)$ by a measurable mapping $f : A^{\mathbb{T}} \to B^{\mathbb{T}}$. The conditional probability defining the channel is

$$\nu_x(G) = \begin{cases} 1 & f(x) \in G \\ 0 & f(x) \notin G. \end{cases}$$

Note that such a channel can also be written as

$$\nu_x(G) = 1_{f^{-1}(G)}(x).$$

A *sequence coder* is a deterministic channel, that is, a measurable mapping from one sequence space into another. It is easy to see that for a deterministic code the hookup is specified by

$$p(F \times G) = \mu(F \bigcap f^{-1}(G))$$

and the output process has distribution

$$\eta(G) = \mu(f^{-1}(G)).$$

A sequence coder is said to be (T_A, T_B)-*stationary* (or just *stationary*) or (T_A^N, T_B^K)-*stationary* (or just (N, K)-*stationary*) if the corresponding channel is. Thus a sequence coder f is stationary if and only if $f(T_A x) = T_B f(x)$ and it is (N, K)- stationary if and only if $f(T_A^N x) = T_B^K f(x)$.

Lemma 2.4. *A stationary deterministic channel is ergodic.*

Proof: From Lemma 2.3 it suffices to show that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} p(T_{A\times B}^{-i}F\bigcap G) = p(F)P(G)$$

for all rectangles of the form $F = F_A \times F_B$, $F_A \in \mathcal{B}_B{}^{\mathbb{T}}$, $F_B \in \mathcal{B}_A{}^{\mathbb{T}}$ and $G = G_A \times G_B$. Then

$$\begin{aligned} p(T_{A\times B}^{-i}F\bigcap G) &= p((T_A^{-i}F_A\bigcap G_A)\times(T_B^{-i}F_B\bigcap G_B)) \\ &= \mu((T_A^{-i}F_A\bigcap G_A)\bigcap f^{-1}(T_B^{-i}F_B\bigcap G_B)). \end{aligned}$$

Since f is stationary and since inverse images preserve set theoretic operations,

$$f^{-1}(T_B^{-i}F_B\bigcap G_B) = T_A^{-i}f^{-1}(F_B)\bigcap f^{-1}(G_B)$$

and hence

$$\begin{aligned} \frac{1}{n}\sum_{i=0}^{n-1} p(T_{A\times B}^{-i}F\bigcap G) &= \frac{1}{n}\sum_{i=0}^{n-1}\mu(T_A^{-i}(F_A\bigcap f^{-1}(F_B))\bigcap G_A\bigcap f^{-1}(G_B)) \\ &\underset{n\to\infty}{\to} \mu(F_A\bigcap f^{-1}(F_B))\mu(G_A\bigcap f^{-1}(G_B)) \\ &= p(F_A\times F_B)p(G_A\times G_B) \end{aligned}$$

since μ is ergodic. This means that the rectangles meet the required condition. Some algebra then will show that finite unions of disjoint sets meeting the conditions also meet the conditions and that complements of sets meeting the conditions also meet them. This implies from the good sets principle (see, for example, p. 14 of [55] or p. 50 in [58]) that the field generated by the rectangles also meets the condition and hence the lemma is proved. □

2.6 Stationary and Sliding-Block Codes

A stationary deterministic channel is also called a *stationary code*, so it follows that the output of a stationary code with a stationary input process is also stationary. A stationary code has a simple and useful structure. Suppose one has a mapping $f : A^{\mathbb{T}} \to B$, that is, a mapping that maps an input sequence into a single output symbol. We can define a complete output sequence y corresponding to an input sequence x by

$$y_n = f(T_A^n x); n \in \mathbb{T}, \tag{2.5}$$

that is, we produce an output, then shift or slide the input sequence by one time unit, and then we produce another output using the same function, and so on. A mapping of this form is called a *sliding-block code*

because it produces outputs by successively sliding an infinite-length input sequence and each time using a fixed mapping to produce the output. The sequence-to-symbol mapping implies a sequence coder, say $\overline{f}$, defined by $\overline{f}(x) = \{f(T_A^n x); n \in \mathbb{T}\}$. Furthermore, $\overline{f}(T_A x) = T_B \overline{f}(x)$, that is, a sliding-block code induces a stationary sequence coder. Conversely, any stationary sequence coder $\overline{f}$ induces a sliding-block code f for which (2.5) holds by the simple identification $f(x) = (\overline{f}(x))_0$, the output at time 0 of the sequence coder. Thus the ideas of stationary sequence coders mapping sequences into sequences and sliding-block codes mapping sequences into letters by sliding the input sequence are equivalent. We can similarly define an (N,K)-sliding-block code which is a mapping $f : A^{\mathbb{T}} \to B^K$ which forms an output sequence y from an input sequence x via the construction

$$y_{nK}^K = f(T_A^{Nn} x).$$

By a similar argument, (N,K)-sliding-block coders are equivalent to (N,K)-stationary sequence coders. When dealing with sliding-block codes we will usually assume for simplicity that K is 1. This involves no loss in generality since it can be made true by redefining the output alphabet.

The following stationarity property of sliding-block codes follows from the properties for stationary channels, but the proof is given for completeness.

Lemma 2.5. *If f is a stationary coding of an AMS process, then the process $\{f_n = fT^n\}$ is also AMS. If the input process is ergodic, then so is $\{f_n\}$.*

Proof: Suppose that the input process has alphabet A_X and distribution P and that the measurement f has alphabet A_f. Define the sequence mapping $\overline{f} : A_X^\infty \to A_f^\infty$ by $\overline{f}(x) = \{f_n(x); n \in \mathbb{T}\}$, where $f_n(x) = f(T^n x)$ and T is the shift on the input sequence space A_X^∞. If T also denotes the shift on the output space, then by construction $\overline{f}(Tx) = T\overline{f}(x)$ and hence for any output event F, $\overline{f}^{-1}(T^{-1}F) = T^{-1}\overline{f}^{-1}(F)$. Let m denote the process distribution for the encoded process. Since $m(F) = P(\overline{f}^{-1}(F))$ for any event $F \in \mathcal{B}(A_f)^\infty$, we have using the stationarity of the mapping f that

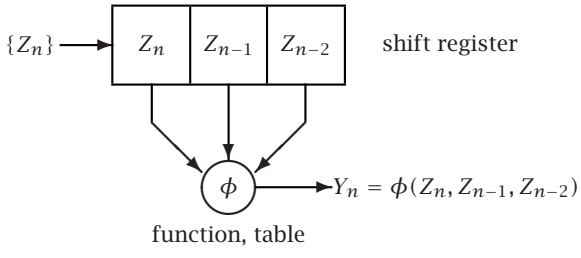
$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} m(T^{-i}F) &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} P(\overline{f}^{-1}(T^{-i}F)) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} P(T^{-i}\overline{f}^{-1}(F)) = \overline{P}(\overline{f}^{-1}(F)),\end{aligned}$$

where $\overline{P}$ is the stationary mean of P. Thus m is AMS. If G is an invariant output event, then $\overline{f}^{-1}(G)$ is also invariant since $T^{-1}\overline{f}^{-1}(G) =$

$\overline{f}^{-1}(T^{-1}G)$. Hence if input invariant sets can only have probability 1 or 0, the same is true for output invariant sets. □

Finite-length Sliding-Block Codes

Stationary or sliding-block codes have a simple description when the sequence-to-symbol mapping characterizing the code depends on only a finite number of the sequence values; that is, the mapping is measurable with respect to a finite number of coordinates. As a particularly simple example, consider the code depicted in Figure 2.1, where an IID process $\{Z_n\}$ consisting of equiprobable coin flips is shifted into a length 3 shift register at the completion of the shift the table is used to produce one output value given the three binary numbers in the shift register. For



$Z_nZ_{n-1}Z_{n-2}$	Y_n
000	0.7683
001	-0.4233
010	-0.1362
011	1.3286
100	0.4233
101	0.1362
110	-1.3286
111	-0.7683

Fig. 2.1 A length 3 stationary code

the curious, this simple code tries to map coin flips into a sequence that looks approximately Gaussian. The output values correspond to eight possible values of an inverse cdf for a 0 mean Gaussian random variable with variance 3/4 evaluated at 8 equally spaced points in the unit interval. The values are "scrambled" to reduce correlation, but the marginal distribution is an approximation to Shannon optimal distribution when simulating or source coding an IID Gaussian sequence with mean 0 and variance 1. All of these ideas will be encountered later in the book.

More generally, suppose that we consider two-sided processes and that we have a measurable mapping

$$\phi : \bigtimes_{i=-M}^{D} A_i \to B$$

and we define a sliding-block code by

$$f(x) = \phi(x_{-M}, \cdots, x_0, \cdots, x_D),$$

so that the output process is

$$Y_n = \phi(X_{n-M}, \cdots, X_n, \cdots, X_{n+D}),$$

a mapping of the contents of a shift register as depicted in Figure 2.2. Note that the time order is reversed in the shift-register representa-

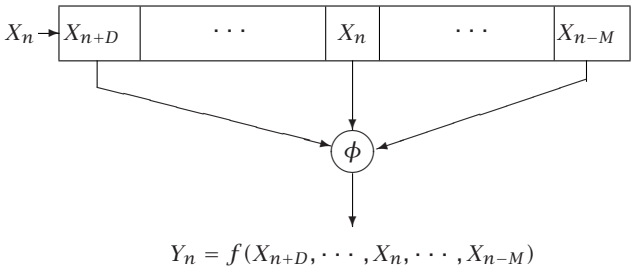


Fig. 2.2 Stationary or sliding-block code

tion since in the shift register new input symbols flow in from the left and exit from the right, but the standard way of writing a sequence is $\cdots, X_{n-2}, X_{n-1}, X_n, X_{n+1}, X_{n+2}, \cdots$ with "past" symbols on the left and "future" symbols on the right. The standard shift is the *left* shift so that shifting the above sequence results in the new sequence $\cdots, X_{n-1}, X_n, X_{n+1}, X_{n+2}, X_{n+3}, \cdots$. Rather than adding to the notational clutter by formally mapping sequences or vectors into a reversed-time form, we shall suffer the minor abuse of notation and follow tradition by using the first format (time increases to the left) for shift-registers, and the second notation (time increases to the right) when dealing with theory and stationary mappings. Context should make the usage clear and clarification will be added when necessary.

The *length* of the code is the length of the shift register or dimension of the vector argument, $L = D + M + 1$.

The mapping ϕ induces a sequence-to-symbol mapping f and a corresponding stationary sequence coder $\overline{f}$. The mapping ϕ is also called a sliding-block code or a finite-length sliding-block code or a finite-window sliding-block code. M is called the *memory* of the code and D is called the *delay* of the code since M past source symbols and D future symbols are required to produce the current output symbol. The *window length* or *constraint length* of the code is $M + D + 1$, the number of input symbols viewed to produce an output symbol. If $D = 0$ the code is said to be *causal*. If $M = 0$ the code is said to be *memoryless*.

There is a problem with the above model if we wish to code a one-sided source since if we start coding at time 0, there are no input symbols with negative indices. Hence we either must require the code be memoryless ($M = 0$) or we must redefine the code for the first M instances (e.g., by "stuffing" the code register with arbitrary symbols) or we must only define the output for times $i \geq M$. For two-sided sources a finite-length sliding-block code is stationary. In the one-sided case it is not even defined precisely unless it is memoryless, in which case it is stationary.

While codes that depend on infinite input sequences may not at first glance seem to be a reasonable physical model of a coding system, it is possible for such codes to depend on the infinite sequence only through a finite number of coordinates. In addition, some real codes may indeed depend on an unboundedly large number of past inputs because of feedback.

Sliding-Block Codes and Partitions

Codes mapping sequences (or vectors) into discrete alphabets have an alternative representation in terms of partitions and range spaces or *codebooks*. Given a sliding-block code $f : A^{\infty} \to B$ where B is discrete, suppose that we index the members of the set B as $B = \{b_i; i \in \mathbb{I}\}$ where $\mathbb{I}$ is a finite or infinite collection of positive integers. Since codes are assumed to be measurable mappings, the sets $P_i = \{x : x \in A^{\infty} : f(x) = b_i\} = f^{-1}(b_i)$, $i \in \mathbb{I}$, collectively form a measurable partition $\mathcal{P} = \{P_i, i \in \mathbb{I}\}$ of A^{∞}; that is, they are disjoint and collectively exhaustive. The sets P_i are referred to as the *atoms* of the partition. The range space $B = \{b_i; i \in \mathbb{I}\}$ is called the *codebook* of the code f or *output alphabet* and it will be assumed without loss of generality that its members are distinct. The code f can be expressed in terms of its partition and codebook by

$$f(x) = \sum_i b_i 1_{P_i}(x), \tag{2.6}$$

where $1_P(x)$ is the indicator function for a set P. Conversely, given a partition and a codebook, (2.6) describes the corresponding code.

B-Processes

One use of sliding-block codes is to provide an easy yet powerful generalization of the simplest class of random processes. IID random pro-

cesses provide the simplest nontrivial example of a random process, typically the first example of a random process encountered in introductory courses is that of sequence of coin flips or rolls of a die. IID processes have no memory and are generally the easiest example to analyze. Stationary or sliding-block coding of an IID process preserves many of the most useful properties of the IID process, including stationarity, ergodicity, and mixing. In addition to providing a common mathematical model of many real processes, processes formed this way turn out to be one of the most important classes of processes in ergodic theory in a way that is relevant to this book — the class of stationary codings of IID processes is exactly the class of random processes for which equal entropy rate is both necessary and sufficient for two processes to be isomorphic in the sense that one can be coded by a stationary code into the other in an invertible way. This result is Ornstein's isomorphism theorem, a result far beyond the scope of this book. But the importance of the class was first recognized in ergodic theory, and adds weight to its emphasis in this presentation of entropy and information theory.

A process is said to be a *B-process* if it can be represented as a finite-alphabet stationary coding of an independent identically distributed (IID) process, where the IID process need not have a finite alphabet. Such processes are also called or *Bernoulli processes* in ergodic theory, but in information theory that name usually implies IID processes (often binary) and not the more general case of any stationary coding of an IID process, so here the name B-process will be used exclusively. The definition also extends to continuous alphabet processes, for example a stationary Gaussian autoregressive processes is also a B-process since it can be represented as the result of passing an IID Gaussian process through a stable autoregressive filter, which is a stationary mapping [173]. The emphasis here, however, will be on ordinary finite-alphabet B-processes. There are many other characterizations of this class of random processes, but the class of stationary codings of IID processes is the simplest and most suitable for the purposes of this book.

Let μ denote the original distribution of the IID process and let η denote the induced output distribution. Then for any output events F and G

$$\eta(F \bigcap T_B^{-n} G) = \mu(\overline{f}^{-1}(F \bigcap T_B^{-n} G)) = \mu(\overline{f}^{-1}(F) \bigcap T_A^{-n} \overline{f}^{-1}(G)),$$

since $\overline{f}$ is stationary. But μ is stationary and mixing since it is IID (see Section 6.7 of [55] or Section 7.7 of [58]) and hence this probability converges to

$$\mu(\overline{f}^{-1}(F))\mu(\overline{f}^{-1}(G)) = \eta(F)\eta(G)$$

and hence η is also mixing. Thus a B-process is mixing of all orders and hence is ergodic with respect to T_B^n for all positive integers n.

B-processes can be thought of as the most random of random processes since they have at their heart an IID process such as coin flips or dice rolls.

2.7 Block Codes

Another case of sequence coding arises when we have a measurable mapping $\alpha : A^N \to B^K$ and we define a sequence coder $f(x) = y$ by

$$y_{nK}^K = (y_{nK}, y_{nK+1}, \cdots, y_{(n+1)K-1}) = \alpha(x_{nN}^N),$$

that is, the input is parsed into nonoverlapping blocks of length N and each is successively coded into a block of length K outputs without regard to past or previous input or output blocks. Clearly N input time units must correspond to K output time units in physical time if the code is to make sense. A code of this form is called a *block code* and it is a special case of an (N, K) sliding block code so that such a code is (T_A^N, T_A^K)-stationary.

Block Independent Processes

As sliding-block coding of an IID process leads to a more general class of random processes, one can also apply a block code to an IID process to obtain a more general class of random processes including IID processes as a special case (with blocklength = 1). The resulting process will be *block independent* in the sense that successive K-blocks will be independent since they depend on independent input N blocks. Unlike B-processes, however, the new processes are not in general stationary or ergodic even if the input was. The process can be modified by inserting a random uniformly distributed start time to convert the K-stationary process into a stationary process, but in general ergodicity is lost and sample functions will still exhibit blocking artifacts.

Sliding-Block vs. Block Codes

We shall be interested in constructing sliding-block codes from block codes and vice versa. Each has its uses. The random process obtained in the next section by sliding-block coding a stationary and ergodic process

will provide a key tool in constructing stationary codes and channels from block-stationary ones.

2.8 Random Punctuation Sequences

This section develops an example of a sliding-block coding of a stationary and ergodic process to obtain a special random process called a random punctuation sequence which can be used to imbed a block structure into a stationary process in a way that preserves ergodicity and mixing properties in codes and channels. Any stationary and ergodic process can be used in the construction, but if the initial process is a B-process, then the resulting puncutation process will also be a B-process.

The results are a a variant of a theorem of Shields and Neuhoff [167] as simplified by Neuhoff and Gilbert [131] for sliding-block codings of finite-alphabet processes.

Lemma 2.6. *Suppose that $\{X_n\}$ is a stationary and ergodic process. Then given N and $\delta > 0$ there exists a stationary (or sliding-block) coding $f : A^{\mathbb{T}} \to \{0, 1, 2\}$ yielding a ternary process $\{Z_n\}$ with the following properties:*

(a) *$\{Z_n\}$ is stationary and ergodic.*

(b) *$\{Z_n\}$ has a ternary alphabet $\{0, 1, 2\}$ and it can output only N-cells of the form $011\cdots1$ (0 followed by $N-1$ ones) or individual 2's. In particular, each 0 is always followed by at exactly $N-1$ 1's.*

(c) *For all integers k*

$$\frac{1-\delta}{N} \le \Pr(Z_k^N = 011\cdots1) \le \frac{1}{N}$$

and hence for any n

$$\Pr(Z_n \text{ is in an } N-\text{cell}) \ge 1-\delta.$$

Comment: A process $\{Z_n\}$ with these properties is called an (N, δ)-*random blocking process* or *punctuation sequence* $\{Z_n\}$. As a visual aid, a segment of a typical punctuation sequence might look like

$$\cdots 111][0\underbrace{11\cdots1}_{N-1\text{'s}}][0\underbrace{11\cdots1}_{N-1\text{'s}}][0\underbrace{11\cdots1}_{N-1\text{'s}}][0\underbrace{11\cdots1}_{N-1\text{'s}}]2[0\underbrace{11\cdots1}_{N-1\text{'s}}][0\underbrace{11\cdots1}_{N-1\text{'s}}]$$
$$[0\underbrace{11\cdots1}_{N-1\text{'s}}][0\underbrace{11\cdots1}_{N-1\text{'s}}][0\underbrace{11\cdots1}_{N-1\text{'s}}][0\underbrace{11\cdots1}_{N-1\text{'s}}][0\underbrace{11\cdots1}_{N-1\text{'s}}]222[0\underbrace{11\cdots1}_{N-1\text{'s}}]0111\cdots$$

with the most of the sequence taken up by N-cells with a few 2's interspersed.

Proof. A sliding-block coding is stationary and hence coding a stationary and ergodic process will yield a stationary and ergodic process (Lemma 2.4), which proves the first part. Pick an $\epsilon > 0$ such that $\epsilon N < \delta$. Given the stationary and ergodic process $\{X_n\}$ (that is also assumed to be aperiodic in the sense that it does not place all of its probability on a finite set of sequences) we can find an event $G \in \mathcal{B}_A{}^{\mathbb{T}}$ having probability less than ϵ. Consider the event $F = G - \bigcup_{i=1}^{N-1} T^{-i}G$, that is, F is the collection of sequences x for which $x \in G$, but $T^i x \notin G$ for $i = 1, \cdots, N-1$. We next develop several properties of this set.

First observe that obviously $\mu(F) \le \mu(G)$ and hence $\mu(F) \le \epsilon$ The sequence of sets $T^{-i}F$ are disjoint since if $y \in T^{-i}F$, then $T^i y \in F \subset G$ and $T^{i+l}y \notin G$ for $l = 1, \cdots, N-1$, which means that $T^j y \notin G$ and hence $T^j y \notin F$ for $N - 1 \ge j > i$. Lastly we need to show that although F may have small probability, it is not 0. To see this suppose the contrary, that is, suppose that $\mu(G - \bigcup_{i=1}^{N-1} T^{-i}G) = 0$. Then

$$\mu(G \bigcap (\bigcup_{i=1}^{N-1} T^{-i}G)) = \mu(G) - \mu(G \bigcap (\bigcup_{i=1}^{N-1} T^{-i}G)^c) = \mu(G)$$

and hence $\mu(\bigcup_{i=1}^{N-1} T^{-i}G|G) = 1$. In words, if G occurs, then it is certain to occur again within the next N shifts. This means that with probability 1 the relative frequency of G in a sequence x must be no less than $1/N$ since if it ever occurs (which it must with probability 1), it must thereafter occur at least once every N shifts. This is a contradiction, however, since this means from the ergodic theorem that $\mu(G) \ge 1/N$ when it was assumed that $\mu(G) \le \epsilon < 1/N$. Thus it must hold that $\mu(F) > 0$.

We now use the rare event F to define a sliding-block code. The general idea is simple, but a more complicated detail will be required to handle a special case. Given a sequence x, define $n(x)$ to be the smallest i for which $T^i x \in F$; that is, we look into the future to find the next occurrence of F. Since F has nonzero probability, $n(x)$ will be finite with probability 1. Intuitively, $n(x)$ should usually be large since F has small probability. Once F is found, we code backwards from that point using blocks of a 0 prefix followed by $N-1$ 1's. The appropriate symbol is then the output of the sliding block code. More precisely, if $n(x) = kN + l$, then the sliding-block code prints a 0 if $l = 0$ and prints a 1 otherwise. This idea suffices until the event F actually occurs at the present time, that is, when $n(x) = 0$. At this point the sliding-block code has just completed printing an N-cell of $0111\cdots1$. It should not automatically start a new N-cell, because at the next shift it will be looking for a new F in the future to code back from and the new cells may not align with the old cells. Thus the coder looks into the future for the next F; that is, it again seeks $n(x)$, the smallest i for which $T^i x \in F$. This time $n(x)$ must be greater than or equal to N since x is now in F and $T^{-i}F$ are disjoint for $i = 1, \cdots N-1$. After finding $n(x) = kN + l$, the coder again codes back

to the origin of time. If $l = 0$, then the two codes are aligned and the coder prints a 0 and continues as before. If $l \neq 0$, then the two codes are not aligned, that is, the current time is in the middle of a new code word. By construction $l \leq N - 1$. In this case the coder prints l 2's (filler poop) and shifts the input sequence l times. At this point there is an $n(x) = kN$ for such that $T^{n(x)}x \in F$ and the coding can proceed as before. Note that k is at least one, that is, there is at least one complete cell before encountering the new F.

By construction, 2's can occur only following the event F and then no more than N 2's can be produced. Thus from the ergodic theorem the relative frequency of 2's (and hence the probability that Z_n is not in an N-block) is no greater than

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_2(Z_0(T^i x)) \leq \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_F(T^i x)N = N\mu(F) \leq N\frac{\delta}{N} = \delta, \tag{2.7}$$

that is,

$$\Pr(Z_n \text{ is in an } N - \text{cell}) \geq 1 - \delta.$$

Since Z_n is stationary by construction,

$$\Pr(Z_k^N = 011\cdots 1) = \Pr(Z_0^N = 011\cdots 1) \text{ for all } k.$$

Thus

$$\Pr(Z_0^N = 011\cdots 1) = \frac{1}{N} \sum_{k=0}^{N-1} \Pr(Z_k^N = 011\cdots 1).$$

The events $\{Z_k^N = 011\cdots 1\}$, $k = 0, 1, \ldots, N-1$ are disjoint, however, since there can be at most one 0 in a single block of N symbols. Thus

$$\begin{aligned} N\Pr(Z^N = 011\cdots 1) &= \sum_{k=0}^{N-1} \Pr(Z_k^N = 011\cdots 1) \\ &= \Pr\Big(\bigcup_{k=0}^{N-1} \{Z_k^N = 011\cdots 1\}\Big). \end{aligned} \tag{2.8}$$

Thus since the rightmost probability is between $1 - \delta$ and 1,

$$\frac{1}{N} \geq \Pr(Z_0^N = 011\cdots 1) \geq \frac{1-\delta}{N}$$

which completes the proof. □

The following corollary shows that a finite-length sliding-block code can be used in the lemma.

Corollary 2.1. *Given the assumptions of the lemma, a finite-length sliding-block code exists with properties (a)-(c).*

Proof. The sets G and hence also F can be chosen in the proof of the lemma to be finite dimensional, that is, to be measurable with respect to $\sigma(X_{-K}, \cdots, X_K)$ for some sufficiently large K. Choose these sets as before with $\delta/2$ replacing δ. Define $n(x)$ as in the proof of the lemma. Since $n(x)$ is finite with probability one, there must be an L such that if

$$B_L = \{x : n(x) > L\},$$

then

$$\mu(B_L) < \frac{\delta}{2}.$$

Modify the construction of the lemma so that if $n(x) > L$, then the sliding-block code prints a 2. Thus if there is no occurrence of the desired finite dimensional pattern in a huge bunch of future symbols, a 2 is produced. If $n(x) < L$, then f is chosen as in the proof of the lemma. The proof now proceeds as in the lemma until (2.7), which is replaced by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_2(Z_0(T^i x)) \le \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_{B_L}(T^i x) + \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_F(T^i x) N \le \delta.$$

The remainder of the proof is the same. □

Application of the lemma to an IID source and merging the symbols 1 and 2 in the punctuation process immediately yield the following result since coding an IID process yields a B-process.

Corollary 2.2. *Given an integer N and a $\delta > 0$ there exists an (N, δ)-punctuation sequence $\{Z_n\}$ with the following properties:*

(a) *$\{Z_n\}$ is B-process (and hence stationary, ergodic, and mixing).*

(b) *$\{Z_n\}$ has a binary alphabet $\{0, 1\}$ and it can output only N-cells of the form $011\cdots1$ (0 followed by $N-1$ ones) or individual ones; that is, each zero is always followed by at least $N-1$ ones.*

(c) *For all integers k*

$$\frac{1-\delta}{N} \le \Pr(Z_k^N = 011\cdots1) \le \frac{1}{N}$$

and hence for any n

$$\Pr(Z_n \text{ is in an } N-\text{cell}) \ge 1-\delta.$$

Random punctuation sequences are closely related to the Rohlin-Kakutani theorem, a classic result of ergodic theory. The language and notation is somewhat different and we shall return to the topic at the end of this chapter.

2.9 Memoryless Channels

Suppose that $q_{x_0}(\cdot)$ is a probability measure on $\mathcal{B}_B$ for all $x_0 \in A$ and that for fixed F ,$q_{x_0}(F)$ is a measurable function of x_0. Let ν be a channel specified by its values on output rectangles by

$$\nu_x(\underset{i\in\mathbb{J}}{\times} F_i) = \prod_{i\in\mathbb{J}} q_{x_i}(F_i),$$

for any finite index set $\mathbb{J} \subset \mathbb{T}$. Then ν is said to be a *memoryless channel*. Intuitively,

$$\Pr(Y_i \in F_i; i \in \mathbb{J}|X) = \prod_{i\in\mathbb{J}} \Pr(Y_i \in F_i|X_i).$$

In fact two forms of memorylessness are evident in a memoryless channel. The channel is *input memoryless* in that the probability of an output event involving $\{Y_i; i \in \{k, k+1, \cdots, m\}\}$ does not involve any inputs before time k, that is, the past inputs. The channel is also *input nonanticipatory* since this event does not depend on inputs after time m, that is, the future inputs. The channel is also *output memoryless* in the sense that for any given input x, output events involving nonoverlapping times are independent, i.e.,

$$\nu_x(Y_1 \in F_1 \bigcap Y_2 \in F_2) = \nu_x(Y_1 \in F_1)\nu_x(Y_2 \in F_2).$$

2.10 Finite-Memory Channels

A channel ν is said to have finite input memory of order M if for all one-sided events F and all n

$$\nu_x((Y_n, Y_{n+1}, \cdots) \in F) = \nu_{x'}((Y_n, Y_{n+1}, \cdots) \in F)$$

whenever $x_i = x'_i$ for $i \geq n - M$. In other words, for an event involving Y_i's after some time n, knowing only the inputs for the same times and M time units earlier completely determines the output probability. Similarly ν is said to have finite anticipation of order L if for all one-sided events F and all n

$$\nu_x((\cdots, Y_n) \in F) = \nu_{x'}((\cdots, Y_n) \in F)$$

provided $x'_i = x_i$ for $i \leq n + L$. That is, at most L future inputs must be known to determine the probability of an event involving current and past outputs.

Channels with finite input memory were introduced by Feinstein [41].

A channel ν is said to have *finite output memory of order K* if for all one-sided events F and G and all inputs x, if $k > K$ then

$$\nu_x((\cdots, Y_n) \in F \bigcap (Y_{n+k}, \cdots) \in G) = \nu_x((\cdots, Y_n) \in F)\nu_x((Y_{n+k}, \cdots) \in G);$$

that is, output events involving output samples separated by more than K time units are independent.

Channels with finite output memory were introduced by Wolfowitz [195].

Channels with finite memory and anticipation are historically important as the first real generalizations of memoryless channels for which coding theorems could be proved. Furthermore, the assumption of finite anticipation is physically reasonable as a model for real-world communication channels. The finite memory assumptions, however, exclude many important examples, e.g., finite-state or Markov channels and channels with feedback filtering action. Hence we will emphasize more general notions which can be viewed as approximations or asymptotic versions of the finite memory assumption. The generalization of finite input memory channels requires some additional tools and is postponed to the next chapter. The notion of finite output memory can be generalized by using the notion of mixing.

2.11 Output Mixing Channels

A channel is said to be output mixing (or asymptotically output independent or asymptotically output memoryless) if for all output rectangles F and G and all input sequences x

$$\lim_{n\to\infty} |\nu_x(T^{-n}F \bigcap G) - \nu_x(T^{-n}F)\nu_x(G)| = 0.$$

More generally it is said to be *output weakly mixing* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\nu_x(T^{-i}F \bigcap G) - \nu_x(T^{-i}F)\nu_x(G)| = 0.$$

Unlike mixing systems, the above definitions for channels place conditions only on output rectangles and not on all output events. Output mixing channels were introduced by Adler [2].

The principal property of output mixing channels is provided by the following lemma.

Lemma 2.7. *If a channel is stationary and output weakly mixing, then it is also ergodic. That is, if ν is stationary and output weakly mixing and if μ is stationary and ergodic, then also $\mu\nu$ is stationary and ergodic.*

Proof: The process $\mu\nu$ is stationary by Lemma 2.1. To prove that it is ergodic it suffices from Lemma 2.3 to prove that for all input/output rectangles of the form $F = F_B \times F_A$, $F_B \in \mathcal{B}_A{}^{\mathbb{T}}$, $F_A \in \mathcal{B}_B{}^{\mathbb{T}}$, and $G = G_B \times G_A$ that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu\nu(T^{-i}F \bigcap G) = \mu\nu(F)\mu\nu(G).$$

We have that

$$\frac{1}{n} \sum_{i=0}^{n-1} \mu\nu(T^{-i}F \bigcap G) - m(F)m(G) =$$

$$\frac{1}{n} \sum_{i=0}^{n-1} \mu\nu((T_B^{-i}F_B \bigcap G_B) \times (T_A^{-i}F_A \bigcap G_A)) - \mu\nu(F_B \times F_A)\mu\nu(G_B \times G_A) =$$

$$\frac{1}{n} \sum_{i=0}^{n-1} \int_{T_A^{-i}F_A \cap G_A} d\mu(x)\nu_x(T_B^{-i}F_B \bigcap G_B) - \mu\nu(F_B \times F_A)\mu(G_B \times G_A) =$$

$$\frac{1}{n} \sum_{i=0}^{n-1} \left(\int_{T_A^{-i}F_A \cap G_A} d\mu(x)\nu_x(T_B^{-i}F_B \bigcap G_B) - \int_{T_A^{-i}F_A \cap G_A} d\mu(x)\nu_x(T_B^{-i}F_B)\nu_x(G_B) \right) +$$

$$\frac{1}{n} \sum_{i=0}^{n-1} \left(\int_{T_A^{-i}F_A \cap G_A} d\mu(x)\nu_x(T_B^{-i}F_B)\nu_x(G_B) - \mu\nu(F_B \times F_A)\mu\nu(G_B \times G_A) \right).$$

The first term is bound above by

$$\frac{1}{n} \sum_{i=0}^{n-1} \int_{T_A^{-i}F_A \cap G_A} d\mu(x) |\nu_x(T_B^{-i}F_B \bigcap G_B) - \nu_x(T_B^{-i}F_B)\nu_x(G_B)| \le$$

$$\int d\mu(x) \frac{1}{n} \sum_{i=0}^{n-1} |\nu_x(T_B^{-i}F_B \bigcap G_B) - \nu_x(T^{-i}F_B)\nu_x(G_B)|$$

which goes to zero from the dominated convergence theorem since the integrand converges to zero from the output weakly mixing assumption. The second term can be expressed using the stationarity of the channel as

$$\int_{F_A} d\mu(x)\nu_x(G_B) \frac{1}{n} \sum_{i=0}^{n-1} 1_{F_A}(T_A^i x)\nu_{T_A^i x}(F_B) - \mu\nu(F)\mu\nu(G).$$

The ergodic theorem implies that as $n \to \infty$ the sample average goes to its expectation

$$\int d\mu(x) 1_{F_A}(x) \nu_x(F_B) = \mu\nu(F)$$

and hence the above formula converges to 0, proving the lemma. □

The lemma provides an example of a completely random channel that is also ergodic in the following corollary.

Corollary 2.3. *Suppose that ν is a stationary completely random channel described by an output measure η. If η is weakly mixing, then ν is ergodic. That is, if μ is stationary and ergodic and η is stationary and weakly mixing, then $\mu\nu = \mu \times \eta$ is stationary and ergodic.*

Proof: If η is weakly mixing, then the channel ν defined by $\nu_x(F) = \eta(F)$, all $x \in A^{\mathbb{T}}$, $F \in \mathcal{B}_B{}^{\mathbb{T}}$ is output weakly mixing. Thus ergodicity follows from the lemma. □

2.12 Block Independent Channels

The idea of a memoryless channel can be extended to a block memoryless or block independent channel. Given integers N and K (usually $K = N$) and a probability measure $q_{x^N}(\cdot)$ on $\mathcal{B}_B^K$ for each $x^N \in A^N$ such that $q_{x^N}(F)$ is a measurable function of x^N for each $F \in \mathcal{B}_B^K$. Let ν be specified by its values on output rectangles by

$$\nu_x(y : y_i \in G_i; i = m, \cdots, m+n-1) = \prod_{i=0}^{\lfloor \frac{n}{K} \rfloor} q_{x_{iN}^N}(G_i),$$

where $G_i \in \mathcal{B}_B$, all i, where $\lfloor z \rfloor$ is the largest integer contained in z, and where

$$G_i = \underset{j=m+iK}{\overset{m+(i+1)K-1}{\times}} F_j$$

with $F_j = B$ if $j \geq m+n$. Such channels are called *block memoryless channels* or *block independent channels*.. A deterministic block independent and block stationary channel is a sequence coder formed by a block code.

The primary use of block independent channels is in the construction of a channel given finite-dimensional conditional probabilities; that is, one has probabilities for output K-tuples given input N-tuples and one wishes to model a channel consistent with these finite-dimensional distributions. The finite-dimensional distributions themselves may be the result of an optimization problem or an estimate based on observed behavior. An immediate problem is that a channel constructed in this

manner may not be stationary, although it is clearly (N,K)-stationary. In Section 2.14 it is seen how to modify a block independent channel so as to produce a stationary channel. The basic idea is to occasionally insert some random spacing between the blocks so as to "stationarize" the channel.

Block independent channels are a special case of the class of conditionally block independent channels, which are considered next.

2.13 Conditionally Block Independent Channels

A *conditionally block independent (CBI)* channel resembles the block independent channel in that for a given input sequence the outputs are block independent. It is more general, however, in that the conditional probabilities of the output block may depend on the entire input sequence (or at least on parts of the input sequence not in the same time block).

A channel is CBI if its values on output rectangles satisfy

$$\nu_x(y : y_i \in F_i; i = m, \cdots, m+n-1) = \prod_{i=0}^{\lfloor \frac{n}{K} \rfloor} \nu_x(y : y_{iN}^N \in G_i).$$

where as before

$$G_i = \bigtimes_{j=m+iK}^{m+(i+1)K-1} F_j$$

with $F_j = B$ if $j \geq m+n$. Block memoryless channels are clearly a special case of CBI channels.

These channels have only finite output memory, but unlike the block independent channels they need not have finite input memory or anticipation.

2.14 Stationarizing Block Independent Channels

Block memoryless channels (and CBI channels) are both block stationary channels. Connecting a stationary input to a block stationary channel will yield a block stationary input/output pair process, but it is sometimes desirable to have a stationary model. In this section we consider a technique of "stationarizing" a block independent channel in order to produce a stationary channel. Intuitively, a stationarized block independent (SBI) channel is a block independent channel with random spacing inserted between the blocks according to a random punctuation process.

That is, when the random blocking process produces N-cells (which is most of the time), the channel uses the N-dimensional conditional distribution. When it is not using an N cell, the channel produces some arbitrary symbol in its output alphabet. We now make this idea precise. Let N, K, and $q_{X^N}(\cdot)$ be as in block independent channel of Section 2.12. We now assume that $K = N$, that is, one output symbol is produced for every input symbol and hence output blocks have the same number of symbols as input blocks. This is done for simplicity as the more general case adds significant notational clutter for minimal conceptual gain.

Given $\delta > 0$ let γ denote the distribution of an (N, δ)-random punctuation sequence $\{Z_n\}$. Let $\mu \times \gamma$ denote the product distribution on $(A^{\mathbb{T}} \times \{0,1\}^{\mathbb{T}}, \mathcal{B}_A^{\mathbb{T}} \times \mathcal{B}_{\{0,1\}}^{\mathbb{T}})$; that is, $\mu \times \gamma$ is the distribution of the pair process $\{X_n, Z_n\}$ consisting of the original source $\{X_n\}$ and the random punctuation source $\{Z_n\}$ with the two sources being independent of one another. Define a regular conditional probability (and hence a channel) $\pi_{x,z}(F)$, $F \in \{\mathcal{B}_B\}^{\mathbb{T}}$, $x \in A^{\mathbb{T}}$, $z \in \{0,1\}^{\mathbb{T}}$ by its values on rectangles as follows: Given z, let $J_2(z)$ denote the collection of indices i for which $z_i = 2$ and hence for which z_i is not in an N-cell and let $J_0(z)$ denote those indices i for which $z_i = 0$, that is, those indices where N-cells begin. Let q^* denote a trivial probability mass function on B placing all of its probability on a reference letter b^*. Given an output rectangle

$$F = \{y : y_j \in F_j;\ j \in \mathbb{J}\} = \underset{j \in \mathbb{J}}{\times} F_j,$$

define

$$\pi_{x,z}(F) = \prod_{i \in \mathbb{J} \bigcap J_2(z)} q^*(F_i) \prod_{i \in \mathbb{J} \bigcap J_0(z)} q_{X_i^N}\Big(\overset{i+N-1}{\underset{j=i}{\times}} F_i\Big),$$

where we assume that $F_i = B$ if $i \notin \mathbb{J}$. Connecting the product source $\mu \times \gamma$ to the channel π yields a hookup process $\{X_n, Z_n, Y_n\}$ with distribution, say, r, which in turn induces a distribution p on the pair process $\{X_n, Y_n\}$ having distribution μ on $\{X_n\}$. If the alphabets are standard, p also induces a regular conditional probability for Y given X and hence a channel ν for which $p = \mu\nu$. A channel of this form is said to be an (N, δ)-*stationarized block independent* or SBI channel.

Lemma 2.8. *An SBI channel is stationary and ergodic. Thus if a stationary (and ergodic) source μ is connected to an SBI channel ν, then the output is stationary (and ergodic).*

Proof: The product source $\mu \times \gamma$ is stationary and the channel π is stationary, hence so is the hookup $(\mu \times \gamma)\pi$ or $\{X_n, Z_n, Y_n\}$. Thus the pair process $\{X_n, Y_n\}$ must also be stationary as claimed. The product source $\mu \times \gamma$ is ergodic from Corollary 2.3 since it can be considered as the input/output process of a completely random channel described by a mixing (hence also weakly mixing) output measure. The channel π is output

strongly mixing by construction and hence is ergodic from Lemma 2.4. Thus the hookup $(\mu \times \gamma)\pi$ must be ergodic. This implies that the coordinate process $\{X_n, Y_n\}$ must also be ergodic. This completes the proof. □

The block independent and SBI channels are useful primarily for proving theorems relating finite-dimensional behavior to sequence behavior and for simulating channels with specified finite-dimensional behavior. The SBI channels will also play a key role in deriving sliding-block coding theorems from block coding theorems by replacing the block distributions by trivial distributions, i.e., by finite-dimensional deterministic mappings or block codes.

The SMB channel was introduced by Pursley and Davisson [29] for finite-alphabet channels and further developed by Gray and Saadat [70], who called it a randomly blocked conditionally independent (RBCI) channel. We opt for the first name because these channels resemble block memoryless channels more than CBI channels.

2.15 Primitive Channels

Primitive channels were introduced by Neuhoff and Shields [136, 133] as a physically motivated general channel model. The idea is that most physical channels combine the input process with a separate noise process that is independent of the signal and then filter the combination in a stationary fashion. The noise is assumed to be IID since the filtering can introduce dependence. The construction of such channels strongly resembles that of the SBI channels. Let γ be the distribution of an IID process $\{Z_n\}$ with alphabet W, let $\mu \times \gamma$ denote the product source formed by an independent joining of the original source distribution μ and the noise process Z_n, let π denote the deterministic channel induced by a stationary sequence coder $f : A^{\mathbb{T}} \times W^{\mathbb{T}} \to B^{\mathbb{T}}$ mapping an input sequence and a noise sequence into an output sequence. Let $r = (\mu \times \gamma)\pi$ denote the resulting hookup distribution and $\{X_n, Z_n, Y_n\}$ denote the resulting process. Let p denote the induced distribution for the pair process $\{X_n, Y_n\}$. If the alphabets are standard, then p and μ together induce a channel $\nu_x(F)$, $x \in A^{\mathbb{T}}$, $F \in \mathcal{B}_B{}^{\mathbb{T}}$. A channel of this form is called a *primitive channel*.

Lemma 2.9. *A primitive channel is stationary with respect to any stationary source and it is ergodic. Thus if μ is stationary and ergodic and ν is primitive, then $\mu\nu$ is stationary and ergodic.*

Proof: Since μ is stationary and ergodic and γ is IID and hence mixing, $\mu \times \nu$ is stationary and ergodic from Corollary 2.3. Since the deterministic channel is stationary, it is also ergodic from Lemma 2.4 and the

resulting triple $\{X_n, Z_n, Y_n\}$ is stationary and ergodic. This implies that the component process $\{X_n, Y_n\}$ must also be stationary and ergodic, completing the proof. □

2.16 Additive Noise Channels

Suppose that $\{X_n\}$ is a source with distribution μ and that $\{W_n\}$ is a "noise" process with distribution ν. Let $\{X_n, W_n\}$ denote the induced product source, that is, the source with distribution $\mu \times \nu$ so that the two processes are independent. Suppose that the two processes take values in a common alphabet A and that A has an addition operation $+$, e.g., it is a semi-group. Define the sliding-block code f by $f(x, w) = x_0 + w_0$ and let $\overline{f}$ denote the corresponding sequence coder. Then as in the primitive channels we have an induced distribution r on triples $\{X_n, W_n, Y_n\}$ and hence a distribution on pairs $\{X_n, Y_n\}$ which with μ induces a channel ν if the alphabets are standard.

Example 2.1. A channel of this form is called a *additive noise channel* or a *signal-independent additive noise channel.*

If the noise process is a B-process, then this is easily seen to be a special case of a primitive channel and hence the channel is stationary with respect to any stationary source and ergodic. If the noise is only known to be stationary, the channel is still stationary with respect to any stationary source. Unless the noise is assumed to be at least weakly mixing, however, it is not known if the channel is ergodic in general.

2.17 Markov Channels

We now consider a special case where A and B are finite sets with the same number of symbols. For a fixed positive integer K, let $\mathbf{P}$ denote the space of all $K \times K$ stochastic matrices $P = \{P(i, j); i, j = 1, 2, \cdots, K\}$. Using the Euclidean metric on this space we can construct the Borel field $\mathcal{P}$ of subsets of $\mathbf{P}$ generated by the open sets to form a measurable space $(\mathbf{P}, \mathcal{P})$. This, in turn, gives a one-sided or two-sided sequence space $(\mathbf{P}^{\mathbb{T}}, \mathcal{P}^{\mathbb{T}})$.

A map $\phi : A^{\mathbb{T}} \to \mathbf{P}^{\mathbb{T}}$ is said to be *stationary* if $\phi T_A = T_P \phi$. Given a sequence $P \in \mathbf{P}^{\mathbb{T}}$, let $\mathcal{M}(P)$ denote the set of all probability measures on $(B^{\mathbb{T}}, \mathcal{B}^{\mathbb{T}})$ with respect to which $Y_m, Y_{m+1}, Y_{m+2}, \cdots$ forms a Markov chain with transition matrices $P_m, P_{m+1}, \cdots$ for any integer m, that is, $\lambda \in \mathcal{M}(P)$ if and only if for any m

$$\lambda[Y_m = y_m, \cdots, Y_n = y_n] = \lambda[Y_m = y_m] \prod_{i=m}^{n-1} P_i(y_i, y_{i+1}),$$
$$n > m, y_m, \cdots, y_n \in B.$$

In the one-sided case only $m = 1$ need be verified. Observe that in general the Markov chain is nonhomogeneous.

A channel $[A, \nu, B]$ is said to be *Markov* if there exists a stationary measurable map $\phi : A^{\mathbb{T}} \to \mathbf{P}^{\mathcal{T}}$ such that $\nu_x \in \mathcal{M}(\phi(x))$, $x \in A^{\mathbb{T}}$.

Markov channels were introduced by Kieffer and Rahe [98] who proved that one-sided and two-sided Markov channels are AMS. Their proof is not included as it is lengthy and involves techniques not otherwise used in this book. The channels are introduced for completeness and to show that several important channels and codes in the literature can be considered as special cases. A variety of conditions for ergodicity for Markov channels are considered in [69]. Most are equivalent to one already considered more generally here: A Markov channel is ergodic if it is output mixing.

2.18 Finite-State Channels and Codes

The most important special cases of Markov channels are finite-state channels and codes. Given a Markov channel with stationary mapping ϕ, the channel is said to be a *finite-state channel (FSC)* if we have a collection of stochastic matrices $P_a \in \mathbf{P}$; $a \in A$ and that $\phi(x)_n = P_{x_n}$, that is, the matrix produced by ϕ at time n depends only on the input at that time, x_n. If the matrices P_a; $a \in A$ contain only 0's and 1's, the channel is called a *finite-state code*. There are several equivalent models of finite-state channels and we pause to consider an alternative form that is more common in information theory. (See Gallager [47], Ch. 4, for a discussion of equivalent models of FSC's and numerous physical examples.) An FSC converts an input sequence x into an output sequence y and a state sequence s according to a conditional probability

$$\Pr(Y_k = y_k, S_k = s_k;\ k = m, \cdots, n | X_i = x_i, S_i = s_i;\ i < m) = \prod_{i=m}^{n} P(y_i, s_i | x_i, s_{i-1}),$$

that is, conditioned on X_i, S_{i-1}, the pair Y_i, S_i is independent of all prior inputs, outputs, and states. This specifies a FSC defined as a special case of a Markov channel where the output sequence above is here the joint state-output sequence $\{y_i, s_i\}$. Note that with this setup, saying the Markov channel is AMS implies that the triple process of source, states,

and outputs is AMS (and hence obviously so is the Gallager input-output process). We will adapt the Kieffer-Rahe viewpoint and call the outputs $\{Y_n\}$ of the Markov channel states even though they may correspond to state-output pairs for a specific physical model.

In the two-sided case, the Markov channel is significantly more general than the FSC because the choice of matrices $\phi(x)_i$ can depend on the past in a very complicated (but stationary) way. One might think that a Markov channel is not a significant generalization of an FSC in the one-sided case, however, because there stationarity of ϕ does not permit a dependence on past channel inputs, only on future inputs, which might seem physically unrealistic. Many practical communications systems do effectively depend on the future, however, by incorporating delay in the coding. The prime example of such look-ahead coders are trellis and tree codes used in an incremental fashion. Such codes investigate many possible output strings several steps into the future to determine the possible effect on the receiver and select the best path, often by a Viterbi algorithm. (See, e.g., Viterbi and Omura [189].) The encoder then outputs only the first symbol of the selected path. While clearly a finite-state machine, this code does not fit the usual model of a finite-state channel or code because of the dependence of the transition matrix on future inputs (unless, of course, one greatly expands the state space). It is, however, a Markov channel.

2.19 Cascade Channels

We will often wish to connect more than one channel in cascade in order to form a communication system, e.g., the original source is connected to a deterministic channel (encoder) which is connected to a communications channel which is in turn connected to another deterministic channel (decoder). We now make precise this idea. Suppose that we are given two channels $[A, \nu^{(1)}, C]$ and $[C, \nu^{(2)}, B]$. The cascade of $\nu^{(1)}$ and $\nu^{(2)}$ is defined as the channel $[A, \nu, B]$ given by

$$\nu_x(F) = \int_{C^{\mathbb{T}}} \nu_u^{(2)}(F)\, d\nu_x^{(1)}(u).$$

In other words, if the original source sequence is X, the output to the first channel and input to the second is U, and the output of the second channel is Y, then $\nu_x^{(1)}(F) = P_{U|X}(F|x)$, $\nu_u(G) = P_{Y|U}(G|u)$, and $\nu_x(G) = P_{Y|X}(G|x)$. Observe that by construction $X \to U \to Y$ is a Markov chain.

Lemma 2.10. *A cascade of two stationary channels is stationary.*

Proof: Let T denote the shift on all of the spaces. Then

$$\nu_x(T^{-1}F) = \int_{C^{\mathbb{T}}} \nu_u^{(2)}(T^{-1}F)d\nu_x^{(1)}(u) = \int_{C^{\mathbb{T}}} \nu_u^{(2)}(F)d\nu_x^{(1)}T^{-1}(u).$$

But $\nu_x^{(1)}(T^{-1}F) = \nu_T x^{(1)}(F)$, that is, the measures $\nu_x^{(1)}T^{-1}$ and $\nu_{Tx}^{(1)}$ are identical and hence the above integral is

$$\int_{C^{\mathbb{T}}} \nu_u^{(2)}(F)\, d\nu_{Tx}^{(1)}(u) = \nu_{Tx}(F),$$

proving the lemma. □

2.20 Communication Systems

A *communication system* consists of a source $[A, \mu]$, a sequence encoder $f : A^{\mathbb{T}} \to B^{\mathbb{T}}$ (a deterministic channel), a channel $[B, \nu, B']$, and a sequence decoder $g : B'^{\mathbb{T}} \to \hat{A}^{\mathbb{T}}$. The overall distribution r is specified by its values on rectangles as

$$r(F_1 \times F_2 \times F_3 \times F_4) = \int_{F_1 \cap f^{-1}(F_2)} d\mu(x)\nu_{f(x)}(F_3 \bigcap g^{-1}(F_4)).$$

Denoting the source by $\{X_n\}$, the encoded source or channel input process by $\{U_n\}$, the channel output process by $\{Y_n\}$, and the decoded process by $\{\hat{X}_n\}$, then r is the distribution of the process $\{X_n, U_n, Y_n, \hat{X}_n\}$. If we let X,U,Y, and $\hat{X}$ denote the corresponding sequences, then observe that $X \to U \to Y$ and $U \to Y \to \hat{X}$ are Markov chains. We abbreviate a communication system to $[\mu, f, \nu, g]$.

It is straightforward from Lemma 2.10 to show that if the source, channel, and coders are stationary, then so is the overall process.

A key topic in information theory, which is a mathematical theory of communication systems, is the characterization of the optimal performance one can obtain for communicating a given source over a given channel using codes within some available class of codes. Precise definitions of *optimal* will be based on the notion of the quality of a system as determined by a measure of *distortion* between input and output to be introduced in Chapter 5.

2.21 Couplings

So far in this chapter the focus has been on combining a source $[A, \mu]$ and a channel $[A, \nu, B]$ or a code which together produce a pair or input/output process $[A \times B, \pi]$, where $\pi = \mu\nu$. The pair process in turn

induces an output process, $[B, \eta]$. Given a pair process $[A \times B, \pi]$, the induced input process $[A, \mu]$ and output process $[B, \eta]$ can be thought of as the *marginal processes* and μ and η the *marginal distributions* of the pair process $[A \times B, \pi]$ and its distribution. From a different viewpoint, we could consider the two marginal processes $[A, \mu]$ and $[B, \eta]$ as being given and define a *coupling* or *joining* of these two processes as any pair process $[A \times B, \pi]$ having the given marginals. Here we can view π as a coupling of μ and η.

In general, given any two processes $[A, \mu]$ and $[B, \eta]$, let $\mathcal{P}(\mu, \eta)$ denote the class of all pair process distributions corresponding to couplings of the two given distributions. This class is not empty because, for example, we can always construct a coupling using product measures. This corresponds to the pair process with the given marginals where the two processes are mutually independent or, in other words, the example of the completely random channel given earlier.

When it is desired to place emphasis on the names of the random processes rather than the distributions, we will refer to a pair process distribution $\pi_{X,Y}$ with marginals π_X and π_Y. If we begin with two separate processes with distributions μ_X and μ_Y, say, then $\mathcal{P}(\mu_X, \mu_Y)$ will denote the collection of all pair processes with marginals $\pi_X = \mu_X$ and $\pi_Y = \mu_Y$. Occasionally $\pi_{X,Y} \in \mathcal{P}(\mu_X, \mu_Y)$ will be abbreviated to $\pi_{X,Y} \Rightarrow \mu_X, \mu_Y$.

If one is given two sources and forms a coupling, then in the case of processes with standard alphabets the coupling implies a channel since the joint process distribution and the input process distribution together imply a conditional distribution of output sequences given input sequences, and this conditional distribution is a regular conditional distribution and hence describes a channel.

Couplings can also be defined for pairs of random vectors rather than random processes in a similar manner.

2.22 Block to Sliding-Block: The Rohlin-Kakutani Theorem

The punctuation sequences of Section 2.14 provide a means for converting a block code into a sliding-block code. Suppose, for example, that $\{X_n\}$ is a source with alphabet A and γ_N is a block code, $\gamma_N : A^N \to B^N$. (The dimensions of the input and output vector are assumed equal to simplify the discussion.) Typically B is binary. As has been argued, block codes are not stationary. One way to stationarize a block code is to use a procedure similar to that used to stationarize a block independent channel: send long sequences of blocks with occasional random spacing to make the overall encoded process stationary. Thus, for example, one could use a sliding-block code to produce a punctuation sequence $\{Z_n\}$ as in Corollary 2.1 which produces isolated 0's followed by KN 1's

and occasionally produces 2's. The sliding-block code uses γ_N to encode a sequence of K source blocks $X_n^N, X_{n+N}^N, \cdots, X_{n+(K-1)N}^N$ if and only if $Z_n = 0$. For those rare times l when $Z_l = 2$, the sliding-block code produces an arbitrary symbol $b^* \in B$. The resulting sliding-block code inherits many of the properties of the original block code, as will be demonstrated when proving theorems for sliding-block codes constructed in this manner. This construction suffices for source coding theorems, but an additional property will be needed when treating the channel coding theorems and other applications. The shortcoming of the results of Lemma 2.6 and Corollary 2.1 is that important source events can depend on the punctuation sequence. In other words, probabilities can be changed by conditioning on the occurrence of $Z_n = 0$ or the beginning of a block code word. In this section we modify the simple construction of Lemma 2.6 to obtain a new punctuation sequence that is approximately independent of certain prespecified events. The result is a variation of the Rohlin-Kakutani theorem of ergodic theory [157] [83]. The development here is patterned after that in Shields [164]. See also Shields and Neuhoff [167].

We begin by recasting the punctuation sequence result in different terms. Given a stationary and ergodic source $\{X_n\}$ with a process distribution μ and a punctuation sequence $\{Z_n\}$ as in Section 2.14, define the set $F = \{x : Z_N(x) = 0\}$, where $x \in A^\infty$ is a two-sided sequence $x = (\cdots, x_{-1}, x_0, x_1, \cdots)$. Let T denote the shift on this sequence space. Restating Corollary 2.1 yields the following.

Lemma 2.11. *Given $\delta > 0$ and an integer N, an L sufficiently large and a set F of sequences that is measurable with respect to $(X_{-L}, \cdots, X_L)$ with the following properties:*

(A) *The sets $T^i F$, $i = 0, 1, \cdots, N-1$ are disjoint.*

(B)

$$\frac{1-\delta}{N} \le \mu(F) \le \frac{1}{N}.$$

(C)

$$1 - \delta \le \mu\Big(\bigcup_{i=0}^{N-1} T^i F\Big).$$

So far all that has been done is to rephrase the punctuation result in more ergodic theory oriented terminology. One can think of the lemma as representing sequence space as a "base" F together with its disjoint shifts $T^i F$; $i = 1, 2, \cdots, N-1$, which make up most of the space, together with whatever is left over, a set $G = A^\infty - \bigcup_{i=0}^{N-1} T^i F$, a set which has probability less than δ which will be called the *garbage set*. This picture is called a *tower* or *Rochlin-Kakutani tower*. The basic construction is pictured in Figure 2.3.

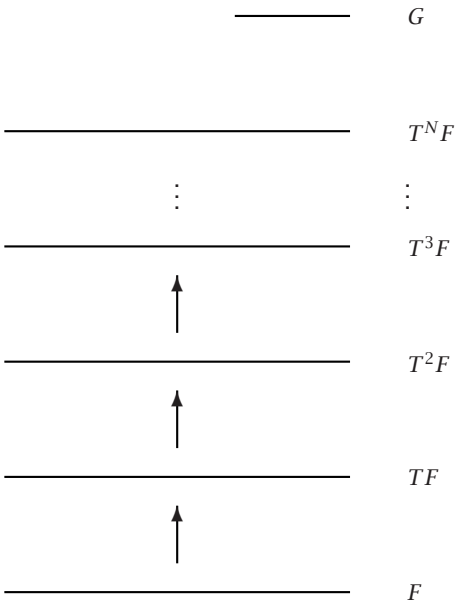


Fig. 2.3 Rohlin-Kakutani Tower

We can relate a tower to a punctuation sequence by identifying the base of the tower, the set F, as the set of sequences of the underlying process which yield $Z_0 = 0$, that is, the punctuation sequence at time 0 yields a 0, indicating the beginning of an N-cell.

Partitions

We now add another wrinkle — consider a finite partition $\mathcal{P} = \{P_i;\ i = 0, 1, \cdots, \|\mathcal{P}\| - 1\}$ of A^∞. One example is the partition of a finite-alphabet sequence space into its possible outputs at time 0, that is, $P_i = \{x : x_0 = a_i\}$ for $i = 0, 1, \cdots, \|A\| - 1$. This is the zero-time partition for the underlying finite-alphabet process. Another possible partition would be according to the output of a sliding-block coding of x, the zero-time partition of the sliding-block coding (or the zero-time partition of the encoded process). In general there is a finite collection of important events that we wish to force to be approximately independent of the punctuation sequence and $\mathcal{P}$ is chosen so that the important events are unions of atoms of $\mathcal{P}$.

Given a partition $\mathcal{P}$, we define the *label* function

$$label_{\mathcal{P}}(x) = \sum_{i=0}^{\|\mathcal{P}\|-1} i1_{P_i}(x),$$

where as usual 1_P is the indicator function of a set P. Thus the label of a sequence is simply the index of the atom of the partition into which it falls.

As $\mathcal{P}$ partitions the input space into which sequences belong to atoms of $\mathcal{P}$, $T^{-i}\mathcal{P}$ partitions the space according to which shifted sequences $T^i x$ belong to atoms of $\mathcal{P}$, that is, $x \in T^{-i}P_l \in T^{-i}\mathcal{P}$ is equivalent to $T^i x \in P_l$ and hence $label_{\mathcal{P}}(T^i x) = l$. The join

$$\mathcal{P}^N = \bigvee_{i=0}^{N-1} T^{-i}\mathcal{P}$$

partitions the space into sequences sharing N labels in the following sense: Each atom Q of $\mathcal{P}^N$ has the form

$$\begin{aligned} Q &= \{x : label_{\mathcal{P}}(x) = k_0, label_{\mathcal{P}}(Tx) = k_1, \cdots, label_{\mathcal{P}}(T^{N-1}x) = k_N - 1\} \\ &= \bigcap_{i=0}^{N-1} T^{-i}P_{k_i} \end{aligned}$$

for some N tuple of integers $\mathbf{k} = (k_0, \cdots, k_N - 1)$. In the ergodic theory literature $\mathbf{k}$ is clled the $\mathcal{P}$-N-*name* of the atom Q. For this reason we index the atoms of $\mathcal{P}^N = \mathcal{Q}$ as $Q_{\mathbf{k}}$. Thus $\mathcal{P}^N$ breaks up the sequence space into groups of sequences which have the same labels for N shifts.

Gadgets

In ergodic theory a *gadget* is a quadruple $(T, F, N, \mathcal{P})$ where T is a transformation (for us a shift), F is an event such that $T^i F; i = 01, \ldots, N-1$ are disjoint (as in a Rohlin-Kakutani tower), and $\mathcal{P}$ is a partition of $\bigcup_{i=0}^{N-1} T^i F$. For concreteness, suppose that $\mathcal{P}$ is the zero-time partition of an underlying process, say a binary IID process. Consider the partition induced in F, the base of the gadget, by $\mathcal{P}^N = \{Q_{\mathbf{k}}\}$, that is, the collection of sets of the form $Q_{\mathbf{k}} \cap F$. By construction, this will be the collection of all infinite sequences for which the punctuation sequence at time zero is 0 ($Z_0 = 0$) and the $\mathcal{P}$-n label of the next N outputs of the process is $\mathbf{k}$, in the binary example there are 2^N such binary N-tuples since $\|\mathcal{P}\| = 2$. The set $Q_{\mathbf{k}} \cap F$ together with its $N-1$ shifts (that is, the set $\bigcup_{i=0}^{N-1} T^i(Q_{\mathbf{k}} \cap F)$) is called a *column* of the gadget.

Gadgets provide an extremely useful structure for using a block code to construct a sliding-block code. Each atom $Q_{\mathbf{k}} \cap F$ in the base partition

contains all sequences corresponding the next N input values being a given binary N-tuple following a punctuation event $Z_0 = 0$.

Strengthened Rohlin-Kakutani Theorem

Lemma 2.12. *Given the assumptions of Lemma 2.11 and a finite partition $\mathcal{P}$, L and F can be chosen so that in addition to properties (A)-(C) it is also true that*

(D)

$$\mu(P_i|F) = \mu(P_i|T^l F);\ l = 1, 2, \cdots, N-1, \tag{2.9}$$

$$\mu(P_i|F) = \mu(P_i|\bigcup_{k=0}^{N-1} T^k F) \tag{2.10}$$

and

$$\mu(P_i \bigcap F) \le \frac{1}{N}\mu(P_i). \tag{2.11}$$

Comment: Eq. (2.11) can be interpreted as stating that P_i and F are approximately independent since $1/N$ is approximately the probability of F. Only the upper bound is stated as it is all we need. Eq. (2.9) also implies that $\mu(P_i \bigcap F)$ is bounded below by $(\mu(P_i) - \delta)\mu(F)$.
Proof: Eq. (2.10) follows from (2.9) since

$$\begin{aligned}
\mu(P_i|\bigcup_{l=0}^{N-1} T^l F) &= \frac{\mu(P_i \bigcap \bigcup_{l=0}^{N-1} T^l F)}{\mu(\bigcup_{l=0}^{N-1} T^l F)} = \frac{\sum_{l=0}^{N-1} \mu(P_i \bigcap T^l F)}{\sum_{l=0}^{N-1} \mu(T^l F)} \\
&= \frac{\sum_{l=0}^{N-1} \mu(P_i|T^l F)\mu(T^l F)}{N\mu(F)} = \frac{1}{N}\sum_{l=0}^{N-1} \mu(P_i|T^l F) \\
&= \mu(P_i \mid F)
\end{aligned}$$

Eq. (2.11) follows from (2.10) since

$$\begin{aligned}
\mu(P_i \bigcap F) &= \mu(P_i|F)\mu(F) = \mu(P_i|\bigcup_{k=0}^{N-1} T^k F)\mu(F) \\
&= \mu(P_i|\bigcup_{k=0}^{N-1} T^k F)\frac{1}{N}\mu(\bigcup_{k=0}^{N-1} T^k F)) \\
&= \frac{1}{N}\mu(P_i \bigcap \bigcup_{k=0}^{N-1} T^k F) \le \frac{1}{N}\mu(P_i)
\end{aligned}$$

since the $T^k F$ are disjoint and have equal probability, The remainder of this section is devoted to proving (2.9).

We first construct using Lemma 2.11 a huge tower of size $KN \gg N$, the height of the tower to be produced for this lemma. Let S denote the base of this original tower and let ϵ by the probability of the garbage set. This height KN tower with base S will be used to construct a new tower of height N and a base F with the additional desired property. First consider the restriction of the partition $\mathcal{P}^N$ to F defined by $\mathcal{P}^N \bigcap F = \{Q_{\mathbf{k}} \bigcap F$; all KN-tuples $\mathbf{k}$ with coordinates taking values in $\{0, 1, \cdots, \|\mathcal{P}\| - 1\}\}$. $\mathcal{P}^N \bigcap F$ divides up the original base according to the labels of NK shifts of base sequences. For each atom $Q_{\mathbf{k}} \bigcap F$ in this base partition, the sets $\{T^l(Q_{\mathbf{k}} \bigcap F);\ k = 0, 1, \cdots, KN - 1\}$ are disjoint and together form a *column* of the tower $\{T^l F;\ k = 0, 1, \cdots, KN - 1\}$. A set of the form $T^l(Q_{\mathbf{k}} \bigcap F)$ is called the lth *level* of the column containing it. Observe that if $\mathcal{Y} \in T^l(Q_{\mathbf{k}} \bigcap F)$, then $\mathcal{Y} = T^l u$ for some $u \in Q_{\mathbf{k}} \bigcap F$ and $T^l u$ has label k_l. Thus we consider k_l to be the label of the column level $T^l(Q_{\mathbf{k}} \bigcap F)$. This complicated structure of columns and levels can be used to recover the original partition by

$$P_j = \bigcup_{l,\mathbf{k}:k_l=j} T^l(Q_{\mathbf{k}} \bigcap F) \bigcap (P_j \bigcap G), \tag{2.12}$$

that is, P_j is the union of all column levels with label j together with that part of P_j in the garbage. We will focus on the pieces of P_j in the column levels as the garbage has very small probability.

We wish to construct a new tower with base F so that the probability of P_i for any of N shifts of F is the same. To do this we form F dividing each column of the original tower into N equal parts. We collect a group of these parts to form F so that F will contain only one part at each level, the N shifts of F will be disjoint, and the union of the N shifts will almost contain all of the original tower. By using the equal probability parts the new base will have conditional probabilities for P_j given T^l equal for all l, as will be shown.

Consider the atom $Q = Q_{\mathbf{k}} \bigcap S$ in the partition $\mathcal{P}^N \bigcap S$ of the base of the original tower. If the source is aperiodic in the sense of placing zero probability on individual sequences, then the set Q can be divided into N disjoint sets of equal probability, say $W_0, W_1, \cdots, W_{N-1}$. Define the set F_Q by

$$\begin{aligned} F_Q &= \Big(\bigcup_{i=0}^{(K-2)N} T^{iN} W_0\Big) \bigcup \Big(\bigcup_{i=0}^{(K-2)N} T^{1+iN} W_1\Big) \bigcup \cdots \Big(\bigcup_{i=0}^{(K-2)N} T^{N-1+iN} W_{N-1}\Big) \\ &= \bigcup_{l=0}^{N-1} \bigcup_{i=0}^{(K-2)N} T^{l+iN} W_l. \end{aligned}$$

F_Q contains $(K-2)$ N shifts of W_0, of $TW_1, \cdots$ of $T^l W_l, \cdots$ and of $T^{N-1} W_{N-1}$. Because it only takes N-shifts of each small set and because

it does not include the top N levels of the original column, shifting F_Q fewer than N times causes no overlap, that is, $T^l F_Q$ are disjoint for $j = 0, 1, \cdots, N-1$. The union of these sets contains all of the original column of the tower except possibly portions of the top and bottom $N-1$ levels (which the construction may not include). The new base F is now defined to be the union of all of the $F_{Q_{\mathbf{k}} \bigcap S}$. The sets $T^l F$ are then disjoint (since all the pieces are) and contain all of the levels of the original tower except possibly the top and bottom $N-1$ levels. Thus

$$\mu(\bigcup_{l=0}^{N-1} T^l F) \geq \mu(\bigcup_{i=N}^{(K-1)N-1} T^i S) = \sum_{i=N}^{(K-1)N-1} \mu(S)$$
$$\geq K - 2\frac{1-\epsilon}{KN} = \frac{1-\epsilon}{N} - \frac{2}{KN}.$$

By choosing $\epsilon = \delta/2$ and K large this can be made larger than $1-\delta$. Thus the new tower satisfies conditions (A)-(C) and we need only verify the new condition (D), that is, (2.9). We have that

$$\mu(P_i | T^l F) = \frac{\mu(P_i \bigcap T^l F)}{\mu(F)}.$$

Since the denominator does not depend on l, we need only show the numerator does not depend on l. From (2.12) applied to the original tower we have that

$$\mu(P_i \bigcap T^l F) = \sum_{j, \mathbf{k}: k_j = i} \mu(T^j(Q_{\mathbf{k}} \bigcap S) \bigcap T^l F),$$

that is, the sum over all column levels (old tower) labeled i of the probability of the intersection of the column level and the lth shift of the new base F. The intersection of a column level in the jth level of the original tower with any shift of F must be an intersection of that column level with the jth shift of one of the sets $W_0, \cdots, W_{N-1}$ (which particular set depends on l). Whichever set is chosen, however, the probability within the sum has the form

$$\mu(T^j(Q_{\mathbf{k}} \bigcap S) \bigcap T^l F) = \mu(T^j(Q_{\mathbf{k}} \bigcap S) \bigcap T^j W_m)$$
$$= \mu((Q_{\mathbf{k}} \bigcap S) \bigcap W_m) = \mu(W_m),$$

where the final step follows since W_m was originally chosen as a subset of $Q_{\mathbf{k}} \bigcap S$. Since these subsets were all chosen to have equal probability, this last probability does not depend on m and hence on l and

$$\mu(T^j(Q_{\mathbf{k}} \bigcap S) \bigcap T^l F) = \frac{1}{N} \mu(Q_{\mathbf{k}} \bigcap S)$$

and hence

$$\mu(P_i \bigcap T^l F) = \sum_{j,\mathbf{k}:k_j=i} \frac{1}{N} \mu(Q_{\mathbf{k}} \bigcap S),$$

which proves (2.9) since there is no dependence on l. This completes the proof of the lemma. □

Chapter 3
Entropy

Abstract The development of the idea of entropy of random variables and processes by Claude Shannon provided the beginnings of information theory and of the modern age of ergodic theory. Entropy and related information measures will be shown to provide useful descriptions of the long term behavior of random processes and this behavior is a key factor in developing the coding theorems of information theory. Here the various notions of entropy for random variables, vectors, processes, and dynamical systems are introduced and and there fundamental properties derived. In this chapter the case of finite-alphabet random processes is emphasized for simplicity, reflecting the historical development of the subject. Occasionally we consider more general cases when it will ease later developments.

3.1 Entropy and Entropy Rate

There are several ways to introduce the notion of entropy and entropy rate. The difference between the two concepts is that entropy is relevant to a single random variable or random vector or, equivalently, to a partition of the sample space, while entropy rate describes a limiting entropy per time unit is we look at sample vectors with increasing dimensions or iterates of a partition. We take some care at the beginning in order to avoid redefining things later. We also try to use definitions resembling the usual definitions of elementary information theory where possible. Let $(\Omega, \mathcal{B}, P, T)$ be a dynamical system. Let $f : \Omega \to A$ be a finite-alphabet measurement (a simple function) defined on Ω and define the random process $f_n = fT^n; n \in \mathbb{T}$. For the moment we focus on one sided processes with $\mathbb{T} = \{0, 1, 2, \ldots\}$. If the transformation T is invertible, we can extend the definition to all integer n and obtain a two-sided process with $\mathbb{T} = \mathbb{Z}$. This process can be viewed as a *coding* of the original space, that

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is, one produces successive coded values by transforming (e.g., shifting) the points of the space, each time producing an output symbol using the same rule or mapping. If Ω is itself a sequence space and T is a shift, then f is a sliding-block code as considered in Section 2.6 and it induces a stationary sequence code $\overline{f} = \{fT^n; n \in \mathbb{T}\}$.

In the usual way we can construct an equivalent Kolmogorov model of this process. Let $A = \{a_1, a_2, \ldots, a_{\|A\|}\}$ denote the finite alphabet of f and let $(A^{Z_+}, \mathcal{B}_A^{Z_+})$ be the resulting one-sided sequence space, where $\mathcal{B}_A$ is the power set. We abbreviate the notation for this sequence space to $(A^\infty, \mathcal{B}_A^\infty)$. Let T_A denote the shift on this space and let X denote the time zero sampling or coordinate function and define $X_n(x) = X(T_A^n x) = x_n$. Let m denote the process distribution induced by the original space and the fT^n, i.e., $m = P_{\overline{f}} = P\overline{f}^{-1}$ where $\overline{f}(\omega) = (f(\omega), f(T\omega), f(T^2\omega), \ldots)$.

Observe that by construction, shifting the input point yields an output sequence that is also shifted, that is,

$$\overline{f}(T\omega) = T_A \overline{f}(\omega).$$

Sequence-valued measurements of this form are called *stationary* or *invariant* codings (or *time-invariant* or *shift-invariant* codings in the case of the shift) since the coding commutes with the transformations. Stationary codes will play an important role throughout this book and are discussed in some detail in Chapter 2. If the input space Ω is itself a sequence space and T is a shift, then the code is also called a *sliding-block code* to reflect the fact that the code operates by shifting the input sequence (sliding) and applying a common measurement or mapping to it. Both the sequence-to-symbol mapping f and the sequence-to-sequence mapping $\overline{f}$ are referred to as a sliding-block code, each implies the other.

The entropy and entropy rates of a finite-alphabet measurement depend only on the process distributions and hence are usually more easily stated in terms of the induced directly given model and the process distribution. For the moment, however, we point out that the definition can be stated in terms of either system. Later we will see that the entropy of the underlying system is defined as a supremum of the entropy rates of all finite-alphabet codings of the system.

The *entropy* of a discrete alphabet random variable f defined on the probability space $(\Omega, \mathcal{B}, P)$ is defined by

$$H_P(f) = -\sum_{a \in A} P(f = a) \ln P(f = a). \tag{3.1}$$

We define $0 \ln 0$ to be 0 in the above formula. We shall often use logarithms to the base 2 instead of natural logarithms. The units for entropy are "nats" when the natural logarithm is used and "bits" for base 2 logarithms. The natural logarithms are usually more convenient for mathe-

matics while the base 2 logarithms provide more intuitive descriptions. The subscript P can be omitted if the measure is clear from context. Be forewarned that the measure will often not be clear from context since more than one measure may be under consideration and hence the subscripts will be required. A discrete alphabet random variable f has a probability mass function (PMF), say p_f, defined by $p_f(a) = P(f = a) = P(\{\omega : f(\omega) = a\})$ and hence we can also write

$$H(f) = -\sum_{a \in A} p_f(a) \ln p_f(a).$$

It is often convenient to consider the entropy not as a function of the particular outputs of f but as a function of the partition that f induces on Ω. In particular, suppose that the alphabet of f is $A = \{a_1, a_2, \ldots, a_{\|A\|}\}$ and define the partition $\mathcal{Q} = \{Q_i; i = 1, 2, \ldots, \|A\|\}$ by $Q_i = \{\omega : f(\omega) = a_i\} = f^{-1}(\{a_i\})$. In other words, $\mathcal{Q}$ consists of disjoint sets which group the points in Ω together according to what output the measurement f produces. We can consider the entropy as a function of the partition and write

$$H_P(\mathcal{Q}) = -\sum_{i=1}^{\|A\|} P(Q_i) \ln P(Q_i). \tag{3.2}$$

Clearly different mappings with different alphabets can have the same entropy if they induce the same partition. Both notations will be used according to the desired emphasis. We have not yet defined entropy for random variables that do not have discrete alphabets; we shall do that later.

Return to the notation emphasizing the mapping f rather than the partition. Defining the random variable $P(f)$ by $P(f)(\omega) = P(\lambda : f(\lambda) = f(\omega))$ we can also write the entropy as

$$H_P(f) = E_P(-\ln P(f)).$$

Using the equivalent directly given model we have immediately that

$$H_P(f) = H_P(\mathcal{Q}) = H_m(X_0) = E_m(-\ln m(X_0)). \tag{3.3}$$

At this point one might ask why we are carrying the baggage of notations for entropy in both the original space and in the sequence space. If we were dealing with only one measurement f (or X_n), we could confine interest to the simpler directly-given form. More generally, however, we will be interested in different measurements or codings on a common system. In this case we will require the notation using the original system. Hence for the moment we keep both forms, but we shall often focus on the second where possible and the first only when necessary.

The *nth order entropy of a discrete alphabet measurement f with respect to T* is defined as

$$H_P^{(n)}(f) = n^{-1}H_P(f^n)$$

where $f^n = (f, fT, fT^2, \ldots, fT^{n-1})$ or, equivalently, we define the discrete alphabet random process $X_n(\omega) = f(T^n\omega)$, then

$$f^n = X^n = X_0, X_1, \ldots, X_{n-1}.$$

As previously, this is given by

$$H_m^{(n)}(X) = n^{-1}H_m(X^n) = n^{-1}E_m(-\ln m(X^n)).$$

This is also called the entropy (per-coordinate or per-sample) of the random vector f^n or X^n. We can also use the partition notation here. The partition corresponding to f^n has a particular form: Suppose that we have two partitions, $\mathcal{Q} = \{Q_i\}$ and $\mathcal{P} = \{P_i\}$. Define their *join* $\mathcal{Q} \bigvee \mathcal{P}$ as the partition containing all nonempty intersection sets of the form $Q_i \bigcap P_j$. Define also $T^{-1}\mathcal{Q}$ as the partition containing the atoms $T^{-1}Q_i$. Then f^n induces the partition

$$\bigvee_{i=0}^{n-1} T^{-i}\mathcal{Q}$$

and we can write

$$H_P^{(n)}(f) = H_P^{(n)}(\mathcal{Q}) = n^{-1}H_P(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{Q}).$$

As before, which notation is preferable depends on whether we wish to emphasize the mapping f or the partition $\mathcal{Q}$.

The *entropy rate* or *mean entropy* of a discrete alphabet measurement f with respect to the transformation T is defined by

$$\begin{aligned}\overline{H}_P(f) &= \limsup_{n\to\infty} H_P^{(n)}(f) \\ &= \overline{H}_P(\mathcal{Q}) = \limsup_{n\to\infty} H_P^{(n)}(\mathcal{Q}) \\ &= \overline{H}_m(X) = \limsup_{n\to\infty} H_m^{(n)}(X).\end{aligned}$$

Given a dynamical system $(\Omega, \mathcal{B}, P, T)$, the *entropy* $H(P,T)$ of the system (or of the measure with respect to the transformation) is defined by

$$H(P,T) = \sup_f \overline{H}_P(f) = \sup_{\mathcal{Q}} \overline{H}_P(\mathcal{Q}), \tag{3.4}$$

where the supremum is over all finite-alphabet measurements (or codings) or, equivalently, over all finite measurable partitions of Ω. (We emphasize that this means alphabets of size M for all finite values of M.) The entropy of a system is also called the *Kolmogorov-Sinai invariant* of the system because of the generalization by Kolmogorov [102] and Sinai [168] of Shannon's entropy rate concept to dynamical systems and the demonstration that equal entropy was a necessary condition for two dynamical systems to be isomorphic.

Note that the entropy rate is well-defined for a continuous-alphabet random process as the supremum over the entropy rates over all finite-alphabet codings of the process. Such an entropy rate is usually infinite, but it is defined.

Suppose that we have a dynamical system corresponding to a finite-alphabet random process $\{X_n\}$, then one possible finite-alphabet measurement on the process is $f(x) = x_0$, that is, the time 0 output. In this case clearly $\overline{H}_P(f) = \overline{H}_P(X)$ and hence, since the system entropy is defined as the supremum over *all* simple measurements,

$$H(P,T) \geq \overline{H}_P(X). \tag{3.5}$$

We shall later see in Theorem 6.1 that (3.5) holds with equality for finite alphabet random processes and provides a generalization of entropy rate for processes that do not have finite alphabets.

3.2 Divergence Inequality and Relative Entropy

Many of the basic properties of entropy follow from a simple result known as the *divergence inequality*. A slight variation is well-known as the log-sum inequality). The divergence or relative entropy is a variation on the idea of entropy and it crops up often as a useful tool for proving and interpreting results and for comparing probability distributions. In this section several fundamental definitions and results are collected together for use in the next section in developing the properties of entropy and entropy rate.

Lemma 3.1. *Given two probability mass functions $\{p_i\}$ and $\{q_i\}$, that is, two countable or finite sequences of nonnegative numbers that sum to one, then*

$$\sum_i p_i \ln \frac{p_i}{q_i} \geq 0$$

with equality if and only if $q_i = p_i$, all i.

Proof: The lemma follows easily from the elementary inequality for real numbers

$$\ln x \le x - 1 \tag{3.6}$$

(with equality if and only if $x = 1$) since

$$\sum_i p_i \ln \frac{q_i}{p_i} \le \sum_i p_i \left(\frac{q_i}{p_i} - 1\right) = \sum_i q_i - \sum_i p_i = 0$$

with equality if and only if $q_i/p_i = 1$ all i. Alternatively, the inequality follows from Jensen's inequality [74] since ln is a convex $\bigcap$ function:

$$\sum_i p_i \ln \frac{q_i}{p_i} \le \ln\left(\sum_i p_i \frac{q_i}{p_i}\right) = 0$$

with equality if and only if $q_i/p_i = 1$, all i. □

The inequality has a simple corollary that we record now for later use.

Corollary 3.1. *The function $x \ln(x/y)$ of real positive x, y is convex in (x, y).*

Proof. Let (x_i, y_i), $i = 1, 2$, be pairs of real positive numbers, $0 \le \lambda \le 1$, and define $x = \lambda x_1 + (1-\lambda)x_2$ and $y = \lambda y_1 + (1-\lambda)y_2$. Apply Lemma 3.1 to the probability mass functions p and q defined by

$$\begin{aligned} p_1 &= \frac{\lambda x_1}{\lambda x_1 + (1-\lambda)x_2} \\ p_2 &= \frac{(1-\lambda)x_2}{\lambda x_1 + (1-\lambda)x_2} \\ q_1 &= \frac{\lambda y_1}{\lambda y_1 + (1-\lambda)y_2} \\ q_2 &= \frac{(1-\lambda)y_2}{\lambda y_1 + (1-\lambda)y_2} \end{aligned}$$

yields

$$0 \le \frac{\lambda x_1}{\lambda x_1 + (1-\lambda)x_2} \ln\left(\frac{\frac{\lambda x_1}{\lambda x_1 + (1-\lambda)x_2}}{\frac{\lambda y_1}{\lambda y_1 + (1-\lambda)y_2}}\right) + \frac{(1-\lambda)x_2}{\lambda x_1 + (1-\lambda)x_2} \ln\left(\frac{\frac{(1-\lambda)x_2}{\lambda x_1 + (1-\lambda)x_2}}{\frac{(1-\lambda)y_2}{\lambda y_1 + (1-\lambda)y_2}}\right).$$

Cancelling the positive denominator and rearranging

$$\lambda x_1 \ln \frac{x_1}{y_1} + (1-\lambda)\lambda x_2 \ln \frac{x_2}{y_2} \ge (\lambda x_1 + (1-\lambda)x_2) \ln\left(\frac{\lambda x_1 + (1-\lambda)x_2}{\lambda y_1 + (1-\lambda)y_2}\right),$$

proving the claimed convexity. □

The quantity used in Lemma 3.1 is of such fundamental importance that we introduce another notion of information and recast the inequality in terms of it. As with entropy, the definition for the moment is only for finite-alphabet random variables. Also as with entropy, there are a variety of ways to define it. Suppose that we have an underlying measurable space $(\Omega, \mathcal{B})$ and two measures on this space, say P and M, and we have a random variable f with finite alphabet A defined on the space and that $\mathcal{Q}$ is the induced partition $\{f^{-1}(a); a \in A\}$. Let P_f and M_f be the induced distributions and let p and m be the corresponding probability mass functions, e.g., $p(a) = P_f(\{a\}) = P(f = a)$. Define the *relative entropy* of a measurement f with measure P with respect to the measure M by

$$H_{P\|M}(f) = H_{P\|M}(\mathcal{Q}) = \sum_{a\in A} p(a) \ln \frac{p(a)}{m(a)} = \sum_{i=1}^{\|A\|} P(Q_i) \ln \frac{P(Q_i)}{M(Q_i)}.$$

Observe that this only makes sense if $p(a)$ is 0 whenever $m(a)$ is, that is, if P_f is absolutely continuous with respect to M_f or $M_f \gg P_f$. Define $H_{P\|M}(f) = \infty$ if P_f is not absolutely continuous with respect to M_f. The measure M is referred to as the *reference measure*. Relative entropies will play an increasingly important role as general alphabets are considered. In the early chapters the emphasis will be on ordinary entropy with similar properties for relative entropies following almost as an afterthought. When considering more abstract (nonfinite) alphabets later on, relative entropies will prove indispensible.

Analogous to entropy, given a random process $\{X_n\}$ described by two process distributions p and m, if it is true that

$$m_{X^n} \gg p_{X^n};\ n = 1, 2, \ldots,$$

then we can define for each n the *nth order relative entropy* $n^{-1}H_{p\|m}(X^n)$ and the *relative entropy rate*

$$\overline{H}_{p\|m}(X) \equiv \limsup_{n\to\infty} \frac{1}{n} H_{p\|m}(X^n).$$

When dealing with relative entropies it is often the measures that are important and not the random variable or partition. We introduce a special notation which emphasizes this fact. Given a probability space $(\Omega, \mathcal{B}, P)$, with Ω a finite space, and another measure M on the same space, we define the *divergence of P with respect to M* as the relative entropy of the identity mapping with respect to the two measures:

$$D(P\|M) = \sum_{\omega\in\Omega} P(\omega) \ln \frac{P(\omega)}{M(\omega)}.$$

Thus, for example, given a finite-alphabet measurement f on an arbitrary probability space $(\Omega, \mathcal{B}, P)$, if M is another measure on $(\Omega, \mathcal{B})$ then

$$H_{P\|M}(f) = D(P_f\|M_f).$$

Similarly,

$$H_{p\|m}(X^n) = D(P_{X^n}\|M_{X^n}),$$

where P_{X^n} and M_{X^n} are the distributions for X^n induced by process measures p and m, respectively. The theory and properties of relative entropy are therefore determined by those for divergence.

There are many names and notations for relative entropy and divergence throughout the literature. The idea was introduced by Kullback for applications of information theory to statistics (see, e.g., Kullback [106] and the references therein) and was used to develop information theoretic results by Perez [145] [147] [146], Dobrushin [32], and Pinsker [150]. Various names in common use for this quantity are discrimination, discrimination information, Kullback-Leibler (KL) number , directed divergence, informational divergence, and cross entropy.

The lemma can be summarized simply in terms of divergence as in the following theorem, which is commonly referred to as the divergence inequality. The assumption of finite alphabets will be dropped later.

Theorem 3.1. Divergence Inequality *Given any two probability measures P and M on a common finite-alphabet probability space, then*

$$D(P\|M) \geq 0 \tag{3.7}$$

with equality if and only if $P = M$.

In this form the result is known as the *divergence inequality*. The fact that the divergence of one probability measure with respect to another is nonnegative and zero only when the two measures are the same suggest the interpretation of divergence as a "distance" between the two probability measures, that is, a measure of how different the two measures are. It is not a true distance or metric in the usual sense since it is not a symmetric function of the two measures and it does not satisfy the triangle inequality. The interpretation is, however, quite useful for adding insight into results characterizing the behavior of divergence and it will later be seen to have implications for ordinary distance measures between probability measures.

The divergence plays a basic role in the family of information measures all of the information measures that we will encounter — entropy, relative entropy, mutual information, and the conditional forms of these information measures — can be expressed as a divergence.

There are three ways to view entropy as a special case of divergence. The first is to permit M to be a general measure instead of requiring it to

be a probability measure and have total mass 1. In this case entropy is minus the divergence if M is the counting measure, i.e., assigns measure 1 to every point in the discrete alphabet. If M is not a probability measure, then the divergence inequality (3.7) need not hold. Second, if the alphabet of f is A_f and has $\|A_f\|$ elements, then letting M be a uniform PMF assigning probability $1/\|A\|$ to all symbols in A yields

$$D(P\|M) = \ln \|A_f\| - H_P(f) \geq 0$$

and hence the entropy is the log of the alphabet size minus the divergence with respect to the uniform distribution. Third, we can also consider entropy a special case of divergence while still requiring that M be a probability measure by using product measures and a bit of a trick. Say we have two measures P and Q on a common probability space $(\Omega, \mathcal{B})$. Define two measures on the product space $(\Omega \times \Omega, \mathcal{B}(\Omega \times \Omega))$ as follows: Let $P \times Q$ denote the usual product measure, that is, the measure specified by its values on rectangles as $P \times Q(F \times G) = P(F)Q(G)$. Thus, for example, if P and Q are discrete distributions with PMF's p and q, then the PMF for $P \times Q$ is just $p(a)q(b)$. Let P' denote the "diagonal" measure defined by its values on rectangles as $P'(F \times G) = P(F \bigcap G)$. In the discrete case P' has PMF $p'(a, b) = p(a)$ if $a = b$ and 0 otherwise. Then

$$H_P(f) = D(P'\|P \times P).$$

Note that if we let X and Y be the coordinate random variables on our product space, then both P' and $P \times P$ give the same marginal probabilities to X and Y, that is, $P_X = P_Y = P$. P' is an extreme distribution on (X, Y) in the sense that with probability one $X = Y$; the two coordinates are deterministically dependent on one another. $P \times P$, however, is the opposite extreme in that it makes the two random variables X and Y independent of one another. Thus the entropy of a distribution P can be viewed as the relative entropy between these two extreme joint distributions having marginals P.

3.3 Basic Properties of Entropy

For the moment fix a probability measure m on a measurable space $(\Omega, \mathcal{B})$ and let X and Y be two finite-alphabet random variables defined on that space. Let A_X and A_Y denote the corresponding alphabets. Let P_{XY}, P_X, and P_Y denote the distributions of (X, Y), X, and Y, respectively.

First observe that since $P_X(a) \leq 1$, all a, $-\ln P_X(a)$ is positive and hence

$$H(X) = -\sum_{a \in A} P_X(a) \ln P_X(a) \geq 0. \tag{3.8}$$

From (3.7) with M uniform as in the second interpretation of entropy above, if X is a random variable with alphabet A_X, then

$$H(X) \le \ln \|A_X\|.$$

Since for any $a \in A_X$ and $b \in A_Y$ we have that $P_X(a) \ge P_{XY}(a,b)$, it follows that

$$\begin{aligned} H(X,Y) &= -\sum_{a,b} P_{XY}(a,b) \ln P_{XY}(a,b) \\ &\ge -\sum_{a,b} P_{XY}(a,b) \ln P_X(a) = H(X). \end{aligned}$$

Using Lemma 3.1 we have that since P_{XY} and $P_X P_Y$ are probability mass functions,

$$H(X,Y) - (H(X) + H(Y)) = \sum_{a,b} P_{XY}(a,b) \ln \frac{P_X(a)P_Y(b)}{P_{XY}(a,b)} \le 0.$$

This proves the following result.

Lemma 3.2. *Given two discrete alphabet random variables X and Y defined on a common probability space, we have*

$$0 \le H(X) \tag{3.9}$$

and

$$\max(H(X), H(Y)) \le H(X,Y) \le H(X) + H(Y) \tag{3.10}$$

where the right hand inequality holds with equality if and only if X and Y are independent. If the alphabet of X has $\|A_X\|$ symbols, then

$$H_X(X) \le \ln \|A_X\|. \tag{3.11}$$

There is another proof of the left hand inequality in (3.10) that uses an inequality for relative entropy that will be useful later when considering codes. The following lemma gives the inequality. First we introduce a definition. A partition $\mathcal{R}$ is said to *refine* a partion $\mathcal{Q}$ if every atom in $\mathcal{Q}$ is a union of atoms of $\mathcal{R}$, in which case we write $\mathcal{Q} < \mathcal{R}$.

Lemma 3.3. *Suppose that P and M are two measures defined on a common measurable space $(\Omega, \mathcal{B})$ and that we are given a finite partitions $\mathcal{Q} < \mathcal{R}$. Then*

$$H_{P\|M}(\mathcal{Q}) \le H_{P\|M}(\mathcal{R})$$

and

$$H_P(\mathcal{Q}) \le H_P(\mathcal{R}).$$

Comments: The lemma can also be stated in terms of random variables and mappings in an intuitive way: Suppose that U is a random variable with finite alphabet A and $f : A \to B$ is a mapping from A into another finite alphabet B. Then the composite random variable $f(U)$ defined by $f(U)(\omega) = f(U(\omega))$ is also a finite alphabet random variable. If U induces a partition $\mathcal{R}$ and $f(U)$ a partition $\mathcal{Q}$, then $\mathcal{Q} < \mathcal{R}$ (since knowing the value of U implies the value of $f(U)$). Thus the lemma immediately gives the following corollary.

Corollary 3.2. *If $M \gg P$ are two measures describing a random variable U with finite alphabet A and if $f : A \to B$, then*

$$H_{P\|M}(f(U)) \le H_{P\|M}(U)$$

and

$$H_P(f(U)) \le H_P(U).$$

Since $D(P_f\|M_f) = H_{P\|M}(f)$, we have also the following corollary which we state for future reference.

Corollary 3.3. *Suppose that P and M are two probability measures on a discrete space and that f is a random variable defined on that space, then*

$$D(P_f\|M_f) \le D(P\|M).$$

The lemma, discussion, and corollaries can all be interpreted as saying that taking a measurement on a finite-alphabet random variable lowers the entropy and the relative entropy of that random variable. By choosing U as (X, Y) and $f(X, Y) = X$ or Y, the lemma yields the promised inequality of the previous lemma.

Proof of Lemma: If $H_{P\|M}(\mathcal{R}) = +\infty$, the result is immediate. If $H_{P\|M}(\mathcal{Q}) = +\infty$, that is, if there exists at least one Q_j such that $M(Q_j) = 0$ but $P(Q_j) \neq 0$, then there exists an $R_i \subset Q_j$ such that $M(R_i) = 0$ and $P(R_i) > 0$ and hence $H_{P\|M}(\mathcal{R}) = +\infty$. Lastly assume that both $H_{P\|M}(\mathcal{R})$ and $H_{P\|M}(\mathcal{Q})$ are finite and consider the difference

$$\begin{aligned} H_{P\|M}(\mathcal{R}) - H_{P\|M}(\mathcal{Q}) &= \sum_i P(R_i) \ln \frac{P(R_i)}{M(R_i)} - \sum_j P(Q_j) \ln \frac{P(Q_j)}{M(Q_j)} \\ &= \sum_j \left[\sum_{i:R_i \subset Q_j} P(R_i) \ln \frac{P(R_i)}{M(R_i)} - P(Q_j) \ln \frac{P(Q_j)}{M(Q_j)} \right]. \end{aligned}$$

We shall show that each of the bracketed terms is nonnegative, which will prove the first inequality. Fix j. If $P(Q_j)$ is 0 we are done since then also $P(R_i)$ is 0 for all i in the inner sum since these R_i all belong to Q_j. If $P(Q_j)$ is not 0, we can divide by it to rewrite the bracketed term as

$$P(Q_j)\left(\sum_{i:R_i\subset Q_j}\frac{P(R_i)}{P(Q_j)}\ln\frac{P(R_i)/P(Q_j)}{M(R_i)/M(Q_j)}\right),$$

where we also used the fact that $M(Q_j)$ cannot be 0 since then $P(Q_j)$ would also have to be zero. Since $R_i\subset Q_j, P(R_i)/P(Q_j)=P(R_i\bigcap Q_j)/P(Q_j)$ $=P(R_i|Q_j)$ is an elementary conditional probability. Applying a similar argument to M and dividing by $P(Q_j)$, the above expression becomes

$$\sum_{i:R_i\subset Q_j}P(R_i|Q_j)\ln\frac{P(R_i|Q_j)}{M(R_i|Q_j)}$$

which is nonnegative from Lemma 3.1, which proves the first inequality. The second inequality follows similarly. Consider the difference

$$\begin{aligned}H_P(\mathcal{R})-H_P(\mathcal{Q}) &= \sum_j\left[\sum_{i:R_i\subset Q_j}P(R_i)\ln\frac{P(Q_j)}{P(R_i)}\right]\\ &= \sum_j P(Q_j)\left[-\sum_{i:R_i\subset Q_j}P(R_i|Q_j)\ln P(R_i|Q_j)\right]\end{aligned}$$

and the result follows since the bracketed term is nonnegative since it is an entropy for each value of j (Lemma 3.2). □

Concavity of Entropy

The next result provides useful inequalities for entropy considered as a function of the underlying distribution. In particular, it shows that entropy is a concave (or convex $\bigcap$) function of the underlying distribution. The concavity follows from Corollary 3.5, but for later use we do a little extra work to obtain an additional property. Define the binary entropy function (the entropy of a binary random variable with probability mass function $(\lambda, 1-\lambda)$) by

$$h_2(\lambda) = -\lambda\ln\lambda-(1-\lambda)\ln(1-\lambda).$$

Lemma 3.4. *Let m and p denote two distributions for a discrete alphabet random variable X and let $\lambda\in(0,1)$. Then for any $\lambda\in(0,1)$*

$$\begin{aligned}\lambda H_m(X)+(1-\lambda)H_p(X) &\le H_{\lambda m+(1-\lambda)p}(X)\\ &\le \lambda H_m(X)+(1-\lambda)H_p(X)+h_2(\lambda).\end{aligned}\quad(3.12)$$

Proof: Define the quantities

$$I = -\sum_x m(x) \ln(\lambda m(x) + (1-\lambda)p(x))$$

and

$$J = H_{\lambda m+(1-\lambda)p}(X) = -\lambda \sum_x m(x) \ln(\lambda m(x) + (1-\lambda)p(x)) - (1-\lambda) \sum_x p(x) \ln(\lambda m(x) + (1-\lambda)p(x)).$$

First observe that

$$\lambda m(x) + (1-\lambda)p(x) \geq \lambda m(x)$$

and therefore applying this bound to both m and p

$$\begin{aligned} I &\leq -\ln\lambda - \sum_x m(x) \ln m(x) = -\ln\lambda + H_m(X) \\ J &\leq -\lambda \sum_x m(x) \ln m(x) - (1-\lambda) \sum_x p(x) \ln p(x) + h_2(\lambda) \\ &= \lambda H_m(X) + (1-\lambda) H_p(X) + h_2(\lambda). \end{aligned} \tag{3.13}$$

To obtain the lower bounds of the lemma observe that

$$\begin{aligned} I &= -\sum_x m(x) \ln m(x) (\lambda + (1-\lambda)\frac{p(x)}{m(x)}) \\ &= -\sum_x m(x) \ln m(x) - \sum_x m(x) \ln(\lambda + (1-\lambda)\frac{p(x)}{m(x)}). \end{aligned}$$

Using (3.6) the rightmost term is bound below by

$$-\sum_x m(x)((\lambda + (1-\lambda)\frac{p(x)}{m(x)} - 1) = -\lambda - 1 + \lambda \sum_{a \in A} p(X=a) + 1 = 0.$$

Thus for all n

$$I \geq -\sum_x m(x) \ln m(x) = H_m(X). \tag{3.14}$$

and hence also

$$\begin{aligned} J &\geq -\lambda \sum_x m(x) \ln m(x) - (1-\lambda) \sum_x p(x) \ln p(x) \\ &= \lambda H_m(X) + (1-\lambda) H_p(X). \end{aligned}$$

□

Convexity of Divergence

Relative entropy possesses a useful convexity property with respect to the two probability measures, as described in the following lemma.

Lemma 3.5. *$D(P\|M)$ is convex in (P, M) for probability measures P, M on a common finite-alphabet probability space, that is, if (P_i, M_i), $i = 1, 2$ are pairs of probability measures, all of which are on a common finite-alphabet probability space, and $(P, M) = \lambda(P_1, M_1) + (1-\lambda)(P_2, M_2)$, then*

$$D(P\|M) \le \lambda D(P_1\|M_1) + (1-\lambda)D(P_2\|M_2).$$

Proof. The result follows from the convexity of $a\ln(a/b)$ in (a, b) from Corollary 3.1. □

Entropy and Binomial Sums

The next result presents an interesting connection between combinatorics and binomial sums with a particular entropy. We require the familiar definition of the binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Lemma 3.6. *Given $\delta \in (0, \frac{1}{2}]$ and a positive integer M, we have*

$$\sum_{i \le \delta M} \binom{M}{i} \le e^{Mh_2(\delta)}. \tag{3.15}$$

If $0 < \delta \le p \le 1$, then

$$\sum_{i \le \delta M} \binom{M}{i} p^i (1-p)^{M-i} \le e^{-Mh_2(\delta\|p)}, \tag{3.16}$$

where

$$h_2(\delta\|p) = \delta \ln \frac{\delta}{p} + (1-\delta)\ln \frac{1-\delta}{1-p}.$$

Proof: We have after some simple algebra that

$$e^{-h_2(\delta)M} = \delta^{\delta M}(1-\delta)^{(1-\delta)M}.$$

If $\delta < 1/2$, then $\delta^k(1-\delta)^{M-k}$ increases as k decreases (since we are having more large terms and fewer small terms in the product) and hence

if $i \le M\delta$,

$$\delta^{\delta M}(1-\delta)^{(1-\delta)M} \le \delta^i(1-\delta)^{M-i}.$$

Thus we have the inequalities

$$1 = \sum_{i=0}^{M} \binom{M}{i} \delta^i(1-\delta)^{M-i} \ge \sum_{i \le \delta M} \binom{M}{i} \delta^i(1-\delta)^{M-i}$$
$$\ge e^{-h_2(\delta)M} \sum_{i \le \delta M} \binom{M}{i}$$

which completes the proof of (3.15). In a similar fashion we have that

$$e^{Mh_2(\delta\|p)} = (\frac{\delta}{p})^{\delta M}(\frac{1-\delta}{1-p})^{(1-\delta)M}.$$

Since $\delta \le p$, we have as in the first argument that for $i \le M\delta$

$$(\frac{\delta}{p})^{\delta M}(\frac{1-\delta}{1-p})^{(1-\delta)M} \le (\frac{\delta}{p})^{i}(\frac{1-\delta}{1-p})^{M-i}$$

and therefore after some algebra we have that if $i \le M\delta$ then

$$p^i(1-p)^{M-i} \le \delta^i(1-\delta)^{M-i}e^{-Mh_2(\delta\|p)}$$

and hence

$$\sum_{i \le \delta M} \binom{M}{i} p^i(1-p)^{M-i} \le e^{-Mh_2(\delta\|p)} \sum_{i \le \delta M} \binom{M}{i} \delta^i(1-\delta)^{M-i}$$
$$\le e^{-nh_2(\delta\|p)} \sum_{i=0}^{M} \binom{M}{i} \delta^i(1-\delta)^{M-i} = e^{-Mh_2(\delta\|p)},$$

which proves (3.16). □

The following is a technical but useful property of sample entropies. The proof follows Billingsley [16].

Lemma 3.7. *Given a finite-alphabet process $\{X_n\}$ (not necessarily stationary) with distribution m, let $X_k^n = (X_k, X_{k+1}, \ldots, X_{k+n-1})$ denote the random vectors giving a block of samples of dimension n starting at time k. Then the random variables $n^{-1} \ln m(X_k^n)$ are m-uniformly integrable (uniform in k and n).*

Proof: For each nonnegative integer r define the sets

$$E_r(k,n) = \{x : -\frac{1}{n} \ln m(x_k^n) \in [r, r+1)\}$$

and hence if $x \in E_r(k,n)$ then

$$r \le -\frac{1}{n} \ln m(x_k^n) < r + 1$$

or

$$e^{-nr} \ge m(x_k^n) > e^{-n(r+1)}.$$

Thus for any r

$$\begin{aligned}\int_{E_r(k,n)} \left(-\frac{1}{n} \ln m(X_k^n)\right) dm &< (r+1) m(E_r(k,n)) \\ &= (r+1) \sum_{x_k^n \in E_r(k,n)} m(x_k^n) \le (r+1) \sum_{x_k^n} e^{-nr} \\ &= (r+1) e^{-nr} \|A\|^n \le (r+1) e^{-nr},\end{aligned}$$

where the final step follows since there are at most $\|A\|^n$ possible n-tuples corresponding to thin cylinders in $E_r(k,n)$ and by construction each has probability less than e^{-nr}.

To prove uniform integrability we must show uniform convergence to 0 as $r \to \infty$ of the integral

$$\begin{aligned}\gamma_r(k,n) &= \int_{x:-\frac{1}{n}\ln m(x_k^n) \ge r} (-\frac{1}{n} \ln m(X_k^n))\, dm = \sum_{i=0}^{\infty} \int_{E_{r+i}(k,n)} (-\frac{1}{n} \ln m(X_k^n))\, dm \\ &\le \sum_{i=0}^{\infty} (r+i+1) e^{-n(r+i)} \|A\|^n \le \sum_{i=0}^{\infty} (r+i+1) e^{-n(r+i-\ln\|A\|)}.\end{aligned}$$

Taking r large enough so that $r > \ln \|A\|$, then the exponential term is bound above by the special case $n = 1$ and we have the bound

$$\gamma_r(k,n) \le \sum_{i=0}^{\infty} (r+i+1) e^{-(r+i-\ln\|A\|)}$$

a bound which is finite and independent of k and n. The sum can easily be shown to go to zero as $r \to \infty$ using standard summation formulas. (The exponential terms shrink faster than the linear terms grow.) □

Variational Description of Divergence

Divergence has a variational characterization that is a fundamental property for its applications to large deviations theory [182] [31]. Although this theory will not be treated here, the basic result of this section provides an alternative description of divergence and hence of relative entropy that has intrinsic interest. The basic result is originally due to Donsker and Varadhan [35].

Suppose now that P and M are two probability measures on a common discrete probability space, say $(\Omega, \mathcal{B})$. Given any real-valued random variable Φ defined on the probability space, we will be interested in the quantity

$$E_M e^{\Phi}. \tag{3.17}$$

which is called the *cumulant generating function* of Φ with respect to M and is related to the characteristic function of the random variable Φ as well as to the moment generating function and the operational transform of the random variable. The following theorem provides a variational description of divergence in terms of the cumulant generating function.

Theorem 3.2.

$$D(P\|M) = \sup_{\Phi} \left(E_P \Phi - \ln(E_M(e^{\Phi})) \right). \tag{3.18}$$

Proof: First consider the random variable Φ defined by

$$\Phi(\omega) = \ln(P(\omega)/M(\omega))$$

and observe that

$$E_P \Phi - \ln(E_M(e^{\Phi})) = \sum_{\omega} P(\omega) \ln \frac{P(\omega)}{M(\omega)} - \ln(\sum_{\omega} M(\omega) \frac{P(\omega)}{M(\omega)})$$

$$= D(P\|M) - \ln 1 = D(P\|M).$$

This proves that the supremum over all Φ is no smaller than the divergence.

To prove the other half observe that for any bounded random variable Φ,

$$E_P \Phi - \ln E_M(e^{\Phi}) = E_P \left(\ln \frac{e^{\Phi}}{E_M(e^{\Phi})} \right) = \sum_{\omega} P(\omega) \left(\ln \frac{M^{\Phi}(\omega)}{M(\omega)} \right),$$

where the probability measure M^{Φ} is defined by

$$M^{\Phi}(\omega) = \frac{M(\omega) e^{\Phi(\omega)}}{\sum_x M(x) e^{\Phi(x)}}.$$

We now have for any Φ that

$$D(P\|Q) - \left(E_P \Phi - \ln(E_M(e^{\Phi})) \right) =$$

$$\sum_{\omega} P(\omega) \left(\ln \frac{P(\omega)}{M(\omega)} \right) - \sum_{\omega} P(\omega) \left(\ln \frac{M^{\Phi}(\omega)}{M(\omega)} \right) =$$

$$\sum_{\omega} P(\omega) \left(\ln \frac{P(\omega)}{M^{\Phi}(\omega)} \right) \geq 0$$

using the divergence inequality. Since this is true for any Φ, it is also true for the supremum over Φ and the theorem is proved. □

3.4 Entropy Rate

For simplicity we focus on the entropy rate of a directly given finite-alphabet random process $\{X_n\}$. We also will emphasize stationary measures, but we will try to clarify those results that require stationarity and those that are more general.

As a reminder, we recall the underlying structure. Let A be a finite set. Let $\Omega = A^{\mathbf{Z}_+}$ and let $\mathcal{B}$ be the sigma-field of subsets of Ω generated by the rectangles. Since A is finite, $(A, \mathcal{B}_A)$ is standard, where $\mathcal{B}_A$ is the power set of A. Thus $(\Omega, \mathcal{B})$ is also standard by Lemma 2.4.1 of [55] or Lemma 3.7 of [58]. In fact, from the proof that cartesian products of standard spaces are standard, we can take as a basis for $\mathcal{B}$ the fields $\mathcal{F}_n$ generated by the finite dimensional rectangles having the form $\{x : X^n(x) = x^n = a^n\}$ for all $a^n \in A^n$ and all positive integers n. (Members of this class of rectangles are called *thin cylinders*.) The union of all such fields, say $\mathcal{F}$, is then a generating field.

Again let $\{X_n; n = 0, 1, \ldots\}$ denote a finite-alphabet random process and apply Lemma 3.2 to vectors and obtain

$$H(X_0, X_1, \ldots, X_{n-1}) \le$$
$$H(X_0, X_1, \ldots, X_{m-1}) + H(X_m, X_{m+1}, \ldots, X_{n-1});\ 0 < m < n. \quad (3.19)$$

Define as usual the random vectors $X_k^n = (X_k, X_{k+1}, \ldots, X_{k+n-1})$, that is, X_k^n is a vector of dimension n consisting of the samples of X from k to $k + n - 1$. If the underlying measure is stationary, then the distributions of the random vectors X_k^n do not depend on k. Hence if we define the sequence $h(n) = H(X^n) = H(X_0, \ldots, X_{n-1})$, then the above equation becomes

$$h(k + n) \le h(k) + h(n);\ \text{all } k, n > 0.$$

Thus $h(n)$ is a subadditive sequence as treated in Section 7.5 of [55] or Section 8.5 of [58]. A basic property of subadditive sequences is that the limit $h(n)/n$ as $n \to \infty$ exists and equals the infimum of $h(n)/n$ over n. (See, e.g., Lemma 7.5.1 of [55] or Lemma 8.5.3 of [58].) This immediately yields the following result.

Lemma 3.8. *If the distribution m of a finite-alphabet random process $\{X_n\}$ is stationary, then*

$$\overline{H}_m(X) = \lim_{n\to\infty} \frac{1}{n} H_m(X^n) = \inf_{n\ge 1} \frac{1}{n} H_m(X^n).$$

Thus the limit exists and equals the infimum.

The next two properties of entropy rate are primarily of interest because they imply a third property, the ergodic decomposition of entropy rate, which will be described in Theorem 3.3. They are also of some independent interest. The first result is a continuity result for entropy rate when considered as a function or functional on the underlying process distribution. The second property demonstrates that entropy rate is actually an affine functional (both convex $\bigcup$ and convex $\bigcap$) of the underlying distribution, even though finite order entropy was only convex $\bigcap$ and not affine.

We apply the distributional distance described in Section 1.7 to the standard sequence measurable space $(\Omega, \mathcal{B}) = (A^{Z_+}, \mathcal{B}_A^{Z_+})$ with a σ-field generated by the countable field $\mathcal{F} = \{F_n; n = 1, 2, \ldots\}$ generated by all thin rectangles.

Corollary 3.4. *The entropy rate $\overline{H}_m(X)$ of a discrete alphabet random process considered as a functional of stationary measures is upper semicontinuous; that is, if probability measures m and m_n, $n = 1, 2, \ldots$ have the property that $d(m, m_n) \to 0$ as $n \to \infty$, then*

$$\overline{H}_m(X) \geq \limsup_{n\to\infty} \overline{H}_{m_n}(X).$$

Proof: For each fixed n

$$H_m(X^n) = -\sum_{a^n \in A^n} m(X^n = a^n) \ln m(X^n = a^n)$$

is a continuous function of m since for the distance to go to zero, the probabilities of all thin rectangles must go to zero and the entropy is the sum of continuous real-valued functions of the probabilities of thin rectangles. Thus we have from Lemma 3.8 that if $d(m_k, m) \to 0$, then

$$\begin{aligned}\overline{H}_m(X) &= \inf_n \frac{1}{n} H_m(X^n) = \inf_n \frac{1}{n} \lim_{k\to\infty} H_{m_k}(X^n) \\ &\geq \limsup_{k\to\infty} \left(\inf_n \frac{1}{n} H_{m_k}(X^n) \right) = \limsup_{k\to\infty} \overline{H}_{m_k}(X).\end{aligned}$$

□

The next lemma uses Lemma 3.4 to show that entropy rates are affine functions of the underlying probability measures.

Lemma 3.9. *Let m and p denote two distributions for a discrete alphabet random process $\{X_n\}$. Then for any $\lambda \in (0, 1)$,*

$$\begin{aligned}\lambda H_m(X^n) + (1-\lambda) H_p(X^n) &\leq H_{\lambda m + (1-\lambda)p}(X^n) \\ &\leq \lambda H_m(X^n) + (1-\lambda) H_p(X^n) + h_2(\lambda), \quad (3.20)\end{aligned}$$

and

$$\limsup_{n\to\infty}(-\int dm(x)\frac{1}{n}\ln(\lambda m(X^n(x)) + (1-\lambda)p(X^n(x))))$$
$$= \limsup_{n\to\infty} -\int dm(x)\frac{1}{n}\ln m(X^n(x)) = \overline{H}_m(X). \quad (3.21)$$

If m and p are stationary then

$$\overline{H}_{\lambda m+(1-\lambda)p}(X) = \lambda\overline{H}_m(X) + (1-\lambda)\overline{H}_p(X) \quad (3.22)$$

and hence the entropy rate of a stationary discrete alphabet random process is an affine function of the process distribution.

Comment: Eq. (3.20) is simply Lemma 3.4 applied to the random vectors X^n stated in terms of the process distributions. Eq. (3.21) states that if we look at the limit of the normalized log of a mixture of a pair of measures when one of the measures governs the process, then the limit of the expectation does not depend on the other measure at all and is simply the entropy rate of the driving source. Thus in a sense the sequences produced by a measure are able to select the true measure from a mixture.

Proof: Eq. (3.20) is just Lemma 3.4. Dividing by n and taking the limit as $n \to \infty$ proves that entropy rate is affine. Similarly, take the limit supremum in expressions (3.13) and (3.14) and the lemma is proved. □

We are now prepared to prove one of the fundamental properties of entropy rate, the fact that it has an ergodic decomposition formula similar to property (c) of Theorem 1.5 when it is considered as a functional on the underlying distribution. In other words, the entropy rate of a stationary source is given by an integral of the entropy rates of the stationary ergodic components. This is a far more complicated result than property (c) of the ordinary ergodic decomposition because the entropy rate depends on the distribution; it is not a simple function of the underlying sequence. The result is due to Jacobs [80].

Theorem 3.3. *The Ergodic Decomposition of Entropy Rate*

Let $(A^{Z_+}, \mathcal{B}(A)^{Z_+}, m, T)$ be a stationary dynamical system corresponding to a stationary finite alphabet source $\{X_n\}$. Let $\{p_x\}$ denote the ergodic decomposition of m. If $\overline{H}_{p_x}(X)$ is m-integrable, then

$$\overline{H}_m(X) = \int dm(x)\overline{H}_{p_x}(X).$$

Proof: The theorem follows immediately from Corollary 3.4 and Lemma 3.9 and the ergodic decomposition of semi-continuous affine funtionals as in Theorem 8.9.1 of [55] or Theorem 8.5 of [58]. □

3.5 Relative Entropy Rate

The properties of relative entropy rate are more difficult to demonstrate. In particular, the obvious analog to (3.19) does not hold for relative entropy rate without the requirement that the reference measure be memoryless, and hence one cannot immediately infer that the relative entropy rate is given by a limit for stationary sources. The following lemma provides a condition under which the relative entropy rate is given by a limit. The condition, that the dominating measure be a kth order (or k-step) Markov source will occur repeatedly when dealing with relative entropy rates. A discrete alphabet source is kth order Markov or k-step Markov (or simply Markov if k is clear from context) if for any n and any $N \geq k$

$$P(X_n = x_n | X_{n-1} = x_{n-1}, \ldots, X_{n-N} = x_{n-N}) \\ = P(X_n = x_n | X_{n-1} = x_{n-1}, \ldots, X_{n-k} = x_{n-k});$$

that is, conditional probabilities given the infinite past depend only on the most recent k symbols. A 0-step Markov source is a memoryless source. A Markov source is said to have *stationary transitions* if the above conditional probabilities do not depend on n, that is, if for any n

$$P(X_n = x_n | X_{n-1} = x_{n-1}, \ldots, X_{n-N} = x_{n-N}) = \\ P(X_k = x_n | X_{k-1} = x_{n-1}, \ldots, X_0 = x_{n-k}).$$

Lemma 3.10. *If p is a stationary process and m is a k-step Markov process with stationary transitions, then*

$$\overline{H}_{p\|m}(X) = \lim_{n\to\infty} \frac{1}{n} H_{p\|m}(X^n) = -\overline{H}_p(X) - E_p[\ln m(X_k|X^k)],$$

where $E_p[\ln m(X_k|X^k)]$ is an abbreviation for

$$E_p[\ln m(X_k|X^k)] = \sum_{x^{k+1}} p_{X^{k+1}}(x^{k+1}) \ln m_{X_k|X^k}(x_k|x^k).$$

Proof: If for any n it is not true that $m_{X^n} \gg p_{X^n}$, then $H_{p\|m}(X^n) = \infty$ for that and all larger n and both sides of the formula are infinite, hence we assume that all of the finite dimensional distributions satisfy the absolute continuity relation. Since m is Markov,

$$m_{X^n}(x^n) = \prod_{l=k}^{n-1} m_{X_l|X^l}(x_l|x^l) m_{X^k}(x^k).$$

Thus

$$\begin{aligned}\frac{1}{n}H_{p\|m}(X^n) &= -\frac{1}{n}H_p(X^n) - \frac{1}{n}\sum_{x^n} p_{X^n}(x^n)\ln m_{X^n}(x^n)\\ &= -\frac{1}{n}H_p(X^n) - \frac{1}{n}\sum_{x^k} p_{X^k}(x^k)\ln m_{X^k}(x^k)\\ &\quad - \frac{n-k}{n}\sum_{x^{k+1}} p_{X^{k+1}}(x^{k+1})\ln m_{X_k|X^k}(x_k|x^k).\end{aligned}$$

Taking limits then yields

$$\overline{H}_{p\|m}(X) = -\overline{H}_p - \sum_{x^{k+1}} p_{X^{k+1}}(x^{k+1})\ln m_{X_k|X^k}(x_k|x^k),$$

where the sum is well defined because if $m_{X_k|X^k}(x_k|x^k) = 0$, then so must $p_{X^{k+1}}(x^{k+1}) = 0$ from absolute continuity. □

Combining the previous lemma with the ergodic decomposition of entropy rate yields the following corollary.

Corollary 3.5. *The Ergodic Decomposition of Relative Entropy Rate*
Let $(A^{Z_+}, \mathcal{B}(A)^{Z_+}, p, T)$ be a stationary dynamical system corresponding to a stationary finite alphabet source $\{X_n\}$. Let m be a kth order Markov process for which $m_{X^n} \gg p_{X^n}$ for all n. Let $\{p_x\}$ denote the ergodic decomposition of p. If $\overline{H}_{p_x\|m}(X)$ is p-integrable, then

$$\overline{H}_{p\|m}(X) = \int dp(x)\overline{H}_{p_x\|m}(X).$$

3.6 Conditional Entropy and Mutual Information

We now turn to other notions of information. While we could do without these if we confined interest to finite-alphabet processes, they will be essential for later generalizations and provide additional intuition and results even in the finite alphabet case. We begin by adding a second finite-alphabet measurement to the setup of the previous sections. To conform more to information theory tradition, we consider the measurements as finite-alphabet random variables X and Y rather than f and g. This has the advantage of releasing f and g for use as functions defined on the random variables: $f(X)$ and $g(Y)$. It should be kept in mind, however, that X and Y could themselves be distinct measurements on a common random variable. This interpretation will often be useful when comparing codes.

Let $(\Omega, \mathcal{B}, P, T)$ be a dynamical system. Let X and Y be finite-alphabet measurements defined on Ω with alphabets A_X and A_Y. Define the *conditional entropy* of X given Y by

$$H(X|Y) \equiv H(X,Y) - H(Y).$$

The name conditional entropy comes from the fact that

$$\begin{aligned} H(X|Y) &= -\sum_{x,y} P(X=a, Y=b) \ln P(X=a|Y=b) \\ &= -\sum_{x,y} p_{X,Y}(x,y) \ln p_{X|Y}(x|y) \\ &= -\sum_{y} p_Y(y) \left[\sum_{x} p_{X|Y}(x \mid y) \ln p_{X|Y}(x|y) \right], \end{aligned}$$

where $p_{X,Y}(x,y)$ is the joint PMF for (X,Y) and

$$p_{X|Y}(x|y) = p_{X,Y}(x,y)/p_Y(y)$$

is the conditional PMF. Defining

$$H(X|Y=y) = -\sum_{x} p_{X|Y}(x|y) \ln p_{X|Y}(x|y)$$

we can also write

$$H(X|Y) = \sum_{y} p_Y(y) H(X|Y=y).$$

Thus conditional entropy is an average of entropies with respect to conditional PMF's. We have immediately from Lemma 3.2 and the definition of conditional entropy that

$$0 \le H(X|Y) \le H(X). \tag{3.23}$$

The inequalities could also be written in terms of the partitions induced by X and Y. Recall that according to Lemma 3.2 the right hand inequality will be an equality if and only if X and Y are independent.

Define the *average mutual information* between X and Y by

$$\begin{aligned} I(X;Y) &= H(X) + H(Y) - H(X,Y) \\ &= H(X) - H(X|Y) = H(Y) - H(Y|X). \end{aligned}$$

In terms of distributions and PMF's we have that

$$\begin{aligned} I(X;Y) &= \sum_{x,y} P(X=x, Y=y) \ln \frac{P(X=x, Y=y)}{P(X=x)P(Y=y)} \\ &= \sum_{x,y} p_{X,Y}(x,y) \ln \frac{p_{X,Y}(x,y)}{p_X(x)p_Y(y)} = \sum_{x,y} p_{X,Y}(x,y) \ln \frac{p_{X|Y}(x|y)}{p_X(x)} \\ &= \sum_{x,y} p_{X,Y}(x,y) \ln \frac{p_{Y|X}(y|x)}{p_Y(y)}. \end{aligned}$$

Note also that mutual information can be expressed as a divergence by

$$I(X;Y) = D(P_{X,Y} \| P_X \times P_Y),$$

where $P_X \times P_Y$ is the product measure on X, Y, that is, a probability measure which gives X and Y the same marginal distributions as P_{XY}, but under which X and Y are independent. Entropy is a special case of mutual information since

$$H(X) = I(X;X).$$

We can collect several of the properties of entropy and relative entropy and produce corresponding properties of mutual information. We state these in the form using measurements, but they can equally well be expressed in terms of partitions.

Lemma 3.11. *Suppose that X and Y are two finite-alphabet random variables defined on a common probability space. Then*

$$0 \le I(X;Y) \le \min(H(X), H(Y)).$$

Suppose that $f : A_X \to A$ and $g : A_Y \to B$ are two measurements. Then

$$I(f(X); g(Y)) \le I(X;Y).$$

Proof: The first result follows immediately from the properties of entropy. The second follows from Lemma 3.3 applied to the measurement (f, g) since mutual information is a special case of relative entropy. □

The next lemma collects some additional, similar properties.

Lemma 3.12. *Given the assumptions of the previous lemma,*

$$\begin{aligned} H(f(X)|X) &= 0, \\ H(X, f(X)) &= H(X), \\ H(X) &= H(f(X)) + H(X|f(X)), \\ I(X; f(X)) &= H(f(X)), \\ H(X|g(Y)) &\geq H(X|Y), \\ I(f(X); g(Y)) &\leq I(X; Y), \\ H(X|Y) &= H(X, f(X, Y))|Y), \end{aligned}$$

and, if Z is a third finite-alphabet random variable defined on the same probability space,

$$H(X|Y) \geq H(X|Y, Z).$$

Comments: The first relation has the interpretation that given a random variable, there is no additional information in a measurement made on the random variable. The second and third relationships follow from the first and the definitions. The third relation is a form of chain rule and it implies that given a measurement on a random variable, the entropy of the random variable is given by that of the measurement plus the conditional entropy of the random variable given the measurement. This provides an alternative proof of the second result of Lemma 3.3. The fifth relation says that conditioning on a measurement of a random variable is less informative than conditioning on the random variable itself. The sixth relation states that coding reduces mutual information as well as entropy. The seventh relation is a conditional extension of the second. The eighth relation says that conditional entropy is nonincreasing when conditioning on more information.

Proof: Since $g(X)$ is a deterministic function of X, the conditional PMF is trivial (a Kronecker delta) and hence $H(g(X)|X = x)$ is 0 for all x, hence the first relation holds. The second and third relations follow from the first and the definition of conditional entropy. The fourth relation follows from the first since $I(X; Y) = H(Y) - H(Y|X)$. The fifth relation follows from the previous lemma since

$$H(X) - H(X|g(Y)) = I(X; g(Y)) \leq I(X; Y) = H(X) - H(X|Y).$$

The sixth relation follows from Corollary 3.3 and the fact that

$$I(X; Y) = D(P_{X,Y} \| P_X \times P_Y).$$

The seventh relation follows since

$$\begin{aligned} H(X, f(X, Y))|Y) &= H(X, f(X, Y)), Y) - H(Y) \\ &= H(X, Y) - H(Y) = H(X|Y). \end{aligned}$$

The final relation follows from the second by replacing Y by Y, Z and setting $g(Y, Z) = Y$. □

In a similar fashion we can consider conditional relative entropies. Suppose now that M and P are two probability measures on a common space, that X and Y are two random variables defined on that space, and that $M_{XY} \gg P_{XY}$ (and hence also $M_X \gg P_Y$). Analagous to the definition of the conditional entropy we can define

$$H_{P\|M}(X|Y) \equiv H_{P\|M}(X, Y) - H_{P\|M}(Y).$$

Some algebra shows that this is equivalent to

$$\begin{aligned} H_{P\|M}(X|Y) &= \sum_{x,y} p_{X,Y}(x, y) \ln \frac{p_{X|Y}(x|y)}{m_{X|Y}(x|y)} \\ &= \sum_{y} p_Y(y) \left(\sum_{x} p_{X|Y}(x|y) \ln \frac{p_{X|Y}(x|y)}{m_{X|Y}(x|y)} \right). \end{aligned} \tag{3.24}$$

This can be written as

$$H_{P\|M}(X|Y) = \sum_{y} p_Y(y) D(p_{X|Y}(\cdot|y) \| m_{X|Y}(\cdot|y)),$$

an average of divergences of conditional PMF's, each of which is well defined because of the original absolute continuity of the joint measure. Manipulations similar to those for entropy can now be used to prove the following properties of conditional relative entropies.

Lemma 3.13. *Given two probability measures M and P on a common space, and two random variables X and Y defined on that space with the property that $M_{XY} \gg P_{XY}$, then the following properties hold:*

$$\begin{aligned} H_{P\|M}(f(X)|X) &= 0, \\ H_{P\|M}(X, f(X)) &= H_{P\|M}(X), \\ H_{P\|M}(X) &= H_{P\|M}(f(X)) + H_{P\|M}(X|f(X)), \end{aligned} \tag{3.25}$$

If $M_{XY} = M_X \times M_Y$ (that is, if the PMFs satisfy $m_{X,Y}(x, y) = m_X(x) m_Y(y)$), then

$$H_{P\|M}(X, Y) \geq H_{P\|M}(X) + H_{P\|M}(Y)$$

and

$$H_{P\|M}(X|Y) \geq H_{P\|M}(X).$$

Eq. (3.25) is a chain rule for relative entropy which provides as a corollary an immediate proof of Lemma 3.3. The final two inequalities resemble inequalities for entropy (with a sign reversal), but they do not hold for all reference measures.

The above lemmas along with Lemma 3.3 show that all of the information measures thus far considered are reduced by taking measurements or by coding. This property is the key to generalizing these quantities to nondiscrete alphabets.

We saw in Lemma 3.4 that entropy was a convex $\bigcap$ function of the underlying distribution. The following lemma provides similar properties of mutual information considered as a function of either a marginal or a conditional distribution.

Lemma 3.14. *Let μ denote a PMF on a discrete space A_X, $\mu(x) = \Pr(X = x)$, and let ν be a conditional PMF, $\nu(y|x) = \Pr(Y = y|X = x)$. Let $\mu\nu$ denote the resulting joint PMF $\mu\nu(x, y) = \mu(x)\nu(y|x)$. Let $I_{\mu\nu} = I_{\mu\nu}(X;Y)$ be the average mutual information. Then $I_{\mu\nu}$ is a convex $\bigcup$ function of ν; that is, given two conditional PMF's ν_1 and ν_2, a $\lambda \in [0,1]$, and $\nu = \lambda\nu_1 + (1-\lambda)\nu_2$, then*

$$I_{\mu\nu} \le \lambda I_{\mu\nu_1} + (1-\lambda) I_{\mu\nu_2},$$

and $I_{\mu\nu}$ is a convex $\bigcap$ function of μ, that is, given two PMF's μ_1 and μ_2, $\lambda \in [0,1]$, and $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$,

$$I_{\mu\nu} \ge \lambda I_{\mu_1\nu} + (1-\lambda) I_{\mu_2\nu}.$$

Proof. First consider the case of a fixed source μ and consider conditional PMFs ν_1, ν_2, and for $0 \le \lambda \le 1$ define $\nu = \lambda\nu_1 + (1-\lambda)\nu_2$. Denote the corresponding input/output pair processes by $p_i = \mu\nu_i$, $i = 1, 2$, and $p = \lambda p_1 + (1-\lambda)p_2$ Let η (respectively, η_1, η_2, η) denote the PMF for Y resulting from ν (respectively ν_1, ν_2, ν), that is, $\eta(y) = \Pr(Y = y) = \sum_x \mu(x)\nu(y|x)$. Note that p_1, p_2, and p all have a common input marginal PMF μ. We have that

$$\mu \times \eta = \lambda\mu \times \eta_1 + (1-\lambda)\mu \times \eta_2$$

so that from Lemma 3.5

$$\begin{aligned} I_{\mu\nu} &= D(\mu\nu||\mu \times \eta) = D(\lambda p_1 + (1-\lambda)p_2 \| \lambda\mu \times \eta_1 + (1-\lambda)\mu \times \eta_2) \\ &\le \lambda D(p_1 \| \mu \times \eta_1) + (1-\lambda) D(p_2 \| \mu \times \eta_2) \\ &= \lambda I_{\mu\nu_1} + (1-\lambda) I_{\mu\nu_2}, \end{aligned}$$

proving the convexity of mutual information with respect to the channel. The author is indebted to T. Linder for suggesting this proof, which is much simpler than the one in the first edition.

Similarly, let $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ and let η_1, η_2, and η denote the corresponding induced output PMF's. Then

$$
\begin{aligned}
I_{\mu\nu} &= \lambda \sum_{x,y} \mu_1(x)\nu(y|x) \log \left(\frac{\nu(y|x)}{\eta(y)} \frac{\eta_1(y)}{\nu(y|x)} \frac{\nu(y|x)}{\eta_1(y)} \right) \\
&\quad + (1-\lambda) \sum_{x,y} \mu_2(x)\nu(y|x) \log \left(\frac{\nu(y|x)}{\eta(y)} \frac{\eta_2(y)}{\nu(y|x)} \frac{\nu(y|x)}{\eta_2(y)} \right) \\
&= \lambda I_{\mu_1\nu} + (1-\lambda) I_{\mu_2\nu} - \lambda \sum_{x,y} \mu_1(x)\nu(y|x) \log \frac{\eta(y)}{\eta_1(y)} \\
&\quad - (1-\lambda) \sum_{x,y} \mu_2(x)\nu(y|x) \log \frac{\eta(y)}{\eta_2(y)} \\
&= \lambda I_{\mu_1\nu} + (1-\lambda) I_{\mu_2\nu} + \lambda D(\eta_1 \| \eta) + (1-\lambda) D(\eta_2 \| \eta) \\
&\geq \lambda I_{\mu_1\nu} + (1-\lambda) I_{\mu_2\nu} \qquad (3.26)
\end{aligned}
$$

from the divergence inequality. □

We consider one other notion of information: Given three finite-alphabet random variables X, Y, Z, define the *conditional mutual information* between X and Y given Z by

$$I(X;Y|Z) = D(P_{XYZ} \| P_{X\times Y|Z}) \qquad (3.27)$$

where $P_{X\times Y|Z}$ is the distribution defined by its values on rectangles as

$$P_{X\times Y|Z}(F \times G \times D) = \sum_{z\in D} P(X \in F|Z=z)P(Y \in G|Z=z)P(Z=z). \qquad (3.28)$$

$P_{X\times Y|Z}$ has the same conditional distributions for X given Z and for Y given Z as does P_{XYZ}, but now X and Y are conditionally independent given Z. Alternatively, the conditional distribution for X, Y given Z under the distribution $P_{X\times Y|Z}$ is the product distribution $P_X|Z \times P_Y|Z$. Thus

$$
\begin{aligned}
I(X;Y|Z) &= \sum_{x,y,z} p_{XYZ}(x,y,z) \ln \frac{p_{XYZ}(x,y,z)}{p_{X|Z}(x|z)p_{Y|Z}(y|z)p_Z(z)} \\
&= \sum_{x,y,z} p_{XYZ}(x,y,z) \ln \frac{p_{XY|Z}(x,y|z)}{p_{X|Z}(x|z)p_{Y|Z}(y|z)}. \qquad (3.29)
\end{aligned}
$$

Since

$$\frac{p_{XYZ}}{p_{X|Z}p_{Y|Z}p_Z} = \frac{p_{XYZ}}{p_X p_{YZ}} \times \frac{p_X}{p_{X|Z}} = \frac{p_{XYZ}}{p_{XZ}p_Y} \times \frac{p_Y}{p_{Y|Z}}$$

we have the first statement in the following lemma.

Lemma 3.15.

$$I(X;Y|Z) + I(Y;Z) = I(Y;(X,Z)), \qquad (3.30)$$

$$I(X;Y|Z) \geq 0, \qquad (3.31)$$

with equality if and only if X and Y are conditionally independent given Z, that is, $p_{XY|Z} = p_{X|Z}p_{Y|Z}$. Given finite valued measurements f and g,

$$I(f(X); g(Y)|Z) \leq I(X; Y|Z).$$

Proof: The second inequality follows from the divergence inequality (3.7) with $P = P_{XYZ}$ and $M = P_{X\times Y|Z}$, i.e., the PMF's p_{XYZ} and $p_{X|Z}p_{Y|Z}p_Z$. The third inequality follows from Lemma 3.3 or its corollary applied to the same measures. □

Comments: Eq. (3.30) is called *Kolmogorov's formula*. If X and Y are conditionally independent given Z in the above sense, then we also have that $p_{X|YZ} = p_{XY|Z}/p_{Y|Z} = p_{X|Z}$, in which case $Y \to Z \to X$ forms a *Markov chain* — given Z, X does not depend on Y. Note that if $Y \to Z \to X$ is a Markov chain, then so is $X \to Z \to Y$. Thus the conditional mutual information is 0 if and only if the variables form a Markov chain with the conditioning variable in the middle. One might be tempted to infer from Lemma 3.3 that given finite valued measurements f, g, and r

$$I(f(X); g(Y)|r(Z)) \overset{(?)}{\leq} I(X; Y|Z).$$

This does not follow, however, since it is not true that if $\mathcal{Q}$ is the partition corresponding to the three quantizers, then $D(P_{f(X),g(Y),r(Z)}\|P_{f(X)\times g(Y)|r(Z)})$ is $H_{P_{X,Y,Z}\|P_{X\times Y|Z}}(f(X), g(Y), r(Z))$ because of the way that $P_{X\times Y|Z}$ is constructed; e.g., the fact that X and Y are conditionally independent given Z implies that $f(X)$ and $g(Y)$ are conditionally independent given Z, but it does not imply that $f(X)$ and $g(Y)$ are conditionally independent given $r(Z)$. Alternatively, if M is $P_{X\times Z|Y}$, then it is not true that $P_{f(X)\times g(Y)|r(Z)}$ equals $M(fgr)^{-1}$. Note that if this inequality were true, choosing $r(z)$ to be trivial (say 1 for all z) would result in $I(X; Y|Z) \geq I(X; Y|r(Z)) = I(X; Y)$. This cannot be true in general since, for example, choosing Z as (X, Y) would give $I(X; Y|Z) = 0$. Thus one must be careful when applying Lemma 3.3 if the measures and random variables are related as they are in the case of conditional mutual information.

We close this section with an easy corollary of the previous lemma and of the definition of conditional entropy. Results of this type are referred to as *chain rules* for information and entropy.

Corollary 3.6. *Given finite-alphabet random variables $Y, X_1, X_2, \ldots, X_n$,*

$$H(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} H(X_i|X_1, \ldots, X_{i-1})$$

$$H_{p\|m}(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} H_{p\|m}(X_i|X_1, \ldots, X_{i-1})$$

$$I(Y;(X_1,X_2,\ldots,X_n)) = \sum_{i=1}^{n} I(Y;X_i|X_1,\ldots,X_{i-1}).$$

3.7 Entropy Rate Revisited

The chain rule of Corollary 3.6 provides a means of computing entropy rates for stationary processes. We have that

$$\frac{1}{n}H(X^n) = \frac{1}{n}\sum_{i=0}^{n-1} H(X_i|X^i).$$

First suppose that the source is a stationary kth order Markov process, that is, for any $m > k$

$$\begin{aligned}\Pr(X_n = x_n|X_i = x_i;\ i = 0,1,\ldots,n-1)\\ = \Pr(X_n = x_n|X_i = x_i;\ i = n-k,\ldots,n-1).\end{aligned}$$

For such a process we have for all $n \geq k$ that

$$H(X_n|X^n) = H(X_n|X_{n-k}^k) = H(X_k|X^k),$$

where $X_i^m = X_i,\ldots,X_{i+m-1}$. Thus taking the limit as $n \to \infty$ of the nth order entropy, all but a finite number of terms in the sum are identical and hence the Cesàro (or arithmetic) mean is given by the conditional expectation. We have therefore proved the following lemma.

Lemma 3.16. *If $\{X_n\}$ is a stationary kth order Markov source, then*

$$\overline{H}(X) = H(X_k|X^k).$$

If we have a two-sided stationary process $\{X_n\}$, then all of the previous definitions for entropies of vectors extend in an obvious fashion and a generalization of the Markov result follows if we use stationarity and the chain rule to write

$$\frac{1}{n}H(X^n) = \frac{1}{n}\sum_{i=0}^{n-1} H(X_0|X_{-1},\ldots,X_{-i}).$$

Since conditional entropy is nonincreasing with more conditioning variables ((3.23) or Lemma 3.12), $H(X_0|X_{-1},\ldots,X_{-i})$ has a limit. Again using the fact that a Cesàro mean of terms all converging to a common limit also converges to the same limit we have the following result.

Lemma 3.17. *If $\{X_n\}$ is a two-sided stationary source, then*

$$\overline{H}(X) = \lim_{n\to\infty} H(X_0|X_{-1},\ldots,X_{-n}).$$

It is tempting to identify the above limit as the conditional entropy given the infinite past, $H(X_0|X_{-1},\ldots)$. Since the conditioning variable is a sequence and does not have a finite alphabet, such a conditional entropy is not included in any of the definitions yet introduced. We shall later demonstrate that this interpretation is indeed valid when the notion of conditional entropy has been suitably generalized.

The natural generalization of Lemma 3.17 to relative entropy rates unfortunately does not work because conditional relative entropies are not in general monotonic with increased conditioning and hence the chain rule does not immediately yield a limiting argument analogous to that for entropy. The argument does work if the reference measure is a kth order Markov, as considered in the following lemma.

Lemma 3.18. *If $\{X_n\}$ is a source described by process distributions p and m and if p is stationary and m is kth order Markov with stationary transitions, then for $n \ge k$ $H_{p\|m}(X_0|X_{-1},\ldots,X_{-n})$ is nondecreasing in n and*

$$\begin{aligned}\overline{H}_{p\|m}(X) &= \lim_{n\to\infty} H_{p\|m}(X_0|X_{-1},\ldots,X_{-n})\\ &= -\overline{H}_p(X) - E_p[\ln m(X_k|X^k)].\end{aligned}$$

Proof: For $n \ge k$ we have that

$$\begin{aligned}&H_{p\|m}(X_0|X_{-1},\ldots,X_{-n}) =\\ &\qquad - H_p(X_0|X_{-1},\ldots,X_{-n}) - \sum_{x^{k+1}} p_{X^{k+1}}(x^{k+1}) \ln m_{X_k|X^k}(x_k|x^k).\end{aligned}$$

Since the conditional entropy is nonincreasing with n and the remaining term does not depend on n, the combination is nondecreasing with n. The remainder of the proof then parallels the entropy rate result. □

It is important to note that the relative entropy analogs to entropy properties often require kth order Markov assumptions on the reference measure (but not on the original measure).

3.8 Markov Approximations

Recall that the relative entropy rate $\overline{H}_{p\|m}(X)$ can be thought of as a distance between the process with distribution p and that with distribution m and that the rate is given by a limit if the reference measure m is Markov. A particular Markov measure relevant to p is the distribution

$p^{(k)}$ which is the *kth order Markov approximation to p* in the sense that it is a kth order Markov source and it has the same kth order transition probabilities as p. To be more precise, the process distribution $p^{(k)}$ is specified by its finite dimensional distributions

$$p^{(k)}_{X^k}(x^k) = p_{X^k}(x^k)$$
$$p^{(k)}_{X^n}(x^n) = p_{X^k}(x^k) \prod_{l=k}^{n-1} p_{X_l|X_{l-k}^k}(x_l|x_{l-k}^k);\ n = k, k+1, \ldots$$

so that

$$p^{(k)}_{X_k|X^k} = p_{X_k|X^k}.$$

It is natural to ask how good this approximation is, especially in the limit, that is, to study the behavior of the relative entropy rate $\overline{H}_{p\|p^{(k)}}(X)$ as $k \to \infty$.

Theorem 3.4. *Given a stationary process p, let $p^{(k)}$ denote the kth order Markov approximations to p. Then*

$$\lim_{k\to\infty} \overline{H}_{p\|p^{(k)}}(X) = \inf_k \overline{H}_{p\|p^{(k)}}(X) = 0.$$

Thus the Markov approximations are asymptotically accurate in the sense that the relative entropy rate between the source and approximation can be made arbitrarily small (zero if the original source itself happens to be Markov).

Proof: As in the proof of Lemma 3.18 we can write for $n \geq k$ that

$$\begin{aligned} &H_{p\|p^{(k)}}(X_0|X_{-1}, \ldots, X_{-n}) \\ &= -H_p(X_0|X_{-1}, \ldots, X_{-n}) - \sum_{x^{k+1}} p_{X^{k+1}}(x^{k+1}) \ln p_{X_k|X^k}(x_k|x^k) \\ &= H_p(X_0|X_{-1}, \ldots, X_{-k}) - H_p(X_0|X_{-1}, \ldots, X_{-n}). \end{aligned}$$

Note that this implies that $p^{(k)}_{X^n} \gg p_{X^n}$ for all n since the entropies are finite. This automatic domination of the finite dimensional distributions of a measure by those of its Markov approximation will not hold in the general case to be encountered later, it is specific to the finite alphabet case. Taking the limit as $n \to \infty$ gives

$$\begin{aligned} \overline{H}_{p\|p^{(k)}}(X) &= \lim_{n\to\infty} H_{p\|p^{(k)}}(X_0|X_{-1}, \ldots, X_{-n}) \\ &= H_p(X_0|X_{-1}, \ldots, X_{-k}) - \overline{H}_p(X). \end{aligned}$$

The corollary then follows immediately from Lemma 3.17. □

Markov approximations will play a fundamental role when considering relative entropies for general (nonfinite-alphabet) processes. The ba-

sic result above will generalize to that case, but the proof will be much more involved.

3.9 Relative Entropy Densities

Many of the convergence results to come will be given and stated in terms of relative entropy densities. In this section we present a simple but important result describing the asymptotic behavior of relative entropy densities. Although the result of this section is only for finite alphabet processes, it is stated and proved in a manner that will extend naturally to more general processes later on. The result will play a fundamental role in the basic ergodic theorems to come.

Throughout this section we will assume that M and P are two process distributions describing a random process $\{X_n\}$. Denote as before the sample vector $X^n = (X_0, X_1, \ldots, X_{n-1})$, that is, the vector beginning at time 0 having length n. The distributions on X^n induced by M and P will be denoted by M_n and P_n, respectively. The corresponding PMF's are m_{X^n} and p_{X^n}. The key assumption in this section is that for all n if $m_{X^n}(x^n) = 0$, then also $p_{X^n}(x^n) = 0$, that is,

$$M_n \gg P_n \text{ for all } n. \tag{3.32}$$

If this is the case, we can define the relative entropy density

$$h_n(x) \equiv \ln \frac{p_{X^n}(x^n)}{m_{X^n}(x^n)} = \ln f_n(x), \tag{3.33}$$

where

$$f_n(x) \equiv \begin{cases} \frac{p_{X^n}(x^n)}{m_{X^n}(x^n)} & \text{if } m_{X^n}(x^n) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.34}$$

Observe that the relative entropy is found by integrating the relative entropy density:

$$\begin{aligned} H_{P\|M}(X^n) = D(P_n\|M_n) &= \sum_{x^n} p_{X^n}(x^n) \ln \frac{p_{X^n}(x^n)}{m_{X^n}(x^n)} \\ &= \int \ln \frac{p_{X^n}(X^n)}{m_{X^n}(X^n)}\, dP \end{aligned} \tag{3.35}$$

Thus, for example, if we assume that

$$H_{P\|M}(X^n) < \infty, \text{ all } n, \tag{3.36}$$

then (3.32) holds.

The following lemma will prove to be useful when comparing the asymptotic behavior of relative entropy densities for different probability measures. It is the first almost everywhere result for relative entropy densities that we consider. It is somewhat narrow in the sense that it only compares limiting densities to zero and not to expectations. We shall later see that essentially the same argument implies the same result for the general case (Theorem 7.4), only the interim steps involving PMF's need be dropped. Note that the lemma requires neither stationarity nor asymptotic mean stationarity.

Lemma 3.19. *Given a finite-alphabet process $\{X_n\}$ with process measures P, M satisfying (3.32), Then*

$$\limsup_{n\to\infty} \frac{1}{n} h_n \le 0, \ M - a.e. \tag{3.37}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} h_n \ge 0, \ P - a.e.. \tag{3.38}$$

If in addition $M \gg P$, then

$$\lim_{n\to\infty} \frac{1}{n} h_n = 0, \ P - a.e.. \tag{3.39}$$

Proof: First consider the probability

$$M(\frac{1}{n} h_n \ge \epsilon) = M(f_n \ge e^{n\epsilon}) \le \frac{E_M(f_n)}{e^{n\epsilon}},$$

where the final inequality is Markov's inequality. But

$$\begin{aligned} E_M(f_n) &= \int dM f_n = \sum_{x^n:\, m_{X^n}(x^n)\neq 0} m_{X^n}(x^n) \frac{p_{X^n}(x^n)}{m_{X^n}(x^n)} \\ &= \sum_{x^n:\, m_{X^n}(x^n)\neq 0} p_{X^n}(x^n) \le 1 \end{aligned}$$

and therefore

$$M(\frac{1}{n} h_n \ge \epsilon) \le 2^{-n\epsilon}$$

and hence

$$\sum_{n=1}^{\infty} M(\frac{1}{n} h_n > \epsilon) \le \sum_{n=1}^{\infty} e^{-n\epsilon} < \infty.$$

From the Borel-Cantelli Lemma (e.g., Lemma 4.6.3 of [55] or Lemma 5.17 of [58]) this implies that $M(n^{-1} h_n \ge \epsilon \text{ i.o.}) = 0$ which implies the first equation of the lemma.

Next consider

$$P(-\frac{1}{n}h_n > \epsilon) = \sum_{x^n:-\frac{1}{n}\ln p_{X^n}(x^n)/m_{X^n}(x^n)>\epsilon} p_{X^n}(x^n)$$
$$= \sum_{x^n:-\frac{1}{n}\ln p_{X^n}(x^n)/m_{X^n}(x^n)>\epsilon \text{ and } m_{X^n}(x^n)\neq 0} p_{X^n}(x^n)$$

where the last statement follows since if $m_{X^n}(x^n) = 0$, then also $p_{X^n}(x^n) = 0$ and hence nothing would be contributed to the sum. In other words, terms violating this condition add zero to the sum and hence adding this condition to the sum does not change the sum's value. Thus

$$P(-\frac{1}{n}h_n > \epsilon)$$
$$= \sum_{x^n:-\frac{1}{n}\ln p_{X^n}(x^n)/m_{X^n}(x^n)>\epsilon \text{ and } m_{X^n}(x^n)\neq 0} \frac{p_{X^n}(x^n)}{m_{X^n}(x^n)} m_{X^n}(x^n)$$
$$= \int_{f_n<e^{-n\epsilon}} dM f_n \leq \int_{f_n<e^{-n\epsilon}} dM e^{-n\epsilon}$$
$$= e^{-n\epsilon} M(f_n < e^{-n\epsilon}) \leq e^{-n\epsilon}.$$

Thus as before we have that $P(n^{-1}h_n > \epsilon) \leq e^{-n\epsilon}$ and hence that $P(n^{-1}h_n \leq -\epsilon \text{ i.o.}) = 0$ which proves the second claim. If also $M \gg P$, then the first equation of the lemma is also true P-a.e., which when coupled with the second equation proves the third. □

Chapter 4
The Entropy Ergodic Theorem

Abstract The goal of this chapter is to prove an ergodic theorem for sample entropy of finite-alphabet random processes. The result is sometimes called the ergodic theorem of information theory or the *asymptotic equipartition (AEP)* theorem, but it is best known as the Shannon-McMillan-Breiman theorem. It provides a common foundation to many of the results of both ergodic theory and information theory.

4.1 History

Shannon [162] first demonstrated the convergence in probability of sample entropy to the entropy rate for stationary ergodic Markov sources. McMillan [123] proved L^1 convergence for stationary ergodic sources and Breiman [20] [21] proved almost everywhere convergence for stationary and ergodic sources. Billingsley [16] extended the result to stationary nonergodic sources. Jacobs [79] [78] extended it to processes dominated by a stationary measure and hence to two-sided AMS processes. Gray and Kieffer [62] extended it to processes asymptotically dominated by a stationary measure and hence to all AMS processes. The generalizations to AMS processes build on the Billingsley theorem for the stationary mean.

Breiman's and Billingsley's approach requires the martingale convergence theorem and embeds the possibly one-sided stationary process into a two-sided process. Ornstein and Weiss [141] developed a proof for the stationary and ergodic case that does not require any martingale theory and considers only positive time and hence does not require any embedding into two-sided processes. The technique was described for both the ordinary ergodic theorem and the entropy ergodic theorem by Shields [165]. In addition, it uses a form of coding argument that is both more direct and more information theoretic in flavor than the traditional martingale proofs. We here follow the Ornstein and Weiss approach for

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the stationary ergodic result. We also use some modifications similar to those of Katznelson and Weiss for the proof of the ergodic theorem. We then generalize the result first to nonergodic processes using the "sandwich" technique of Algoet and Cover [7] and then to AMS processes using a variation on a result of [62].

We next state the theorem to serve as a guide through the various steps. We also prove the result for the simple special case of a Markov source, for which the result follows from the usual ergodic theorem.

We consider a directly given finite-alphabet source $\{X_n\}$ described by a distribution m on the sequence measurable space $(\Omega, \mathcal{B})$. Define as previously $X_k^n = (X_k, X_{k+1}, \cdots, X_{k+n-1})$. The subscript is omitted when it is zero. For any random variable Y defined on the sequence space (such as X_k^n) we define the random variable $m(Y)$ by $m(Y)(x) = m(Y = Y(x))$.

Theorem 4.1. *The Entropy Ergodic Theorem*

Given a finite-alphabet AMS source $\{X_n\}$ with process distribution m and stationary mean $\overline{m}$, let $\{\overline{m}_x; x \in \Omega\}$ be the ergodic decomposition of the stationary mean $\overline{m}$. Then

$$\lim_{n\to\infty} \frac{-\ln m(X^n)}{n} = h; \ m - a.e. \text{ and in } L^1(m), \tag{4.1}$$

where $h(x)$ is the invariant function defined by

$$h(x) = \overline{H}_{\overline{m}_x}(X). \tag{4.2}$$

Furthermore,

$$E_m h = \lim_{n\to\infty} \frac{1}{n} H_m(X^n) = \overline{H}_m(X); \tag{4.3}$$

that is, the entropy rate of an AMS process is given by the limit, and

$$\overline{H}_{\overline{m}}(X) = \overline{H}_m(X). \tag{4.4}$$

Comments: The theorem states that the sample entropy using the AMS measure m converges to the entropy rate of the underlying ergodic component of the stationary mean. Thus, for example, if m is itself stationary and ergodic, then the sample entropy converges to the entropy rate of the process m-a.e. and in $L^1(m)$. The $L^1(m)$ convergence follows immediately from the almost everywhere convergence and the fact that sample entropy is uniformly integrable (Lemma 3.7). L^1 convergence in turn immediately implies the left-hand equality of (4.3). Since the limit exists, it is the entropy rate. The final equality states that the entropy rates of an AMS process and its stationary mean are the same. This result follows from (4.2)-(4.3) by the following argument: We have that $\overline{H}_m(X) = E_m h$ and $\overline{H}_{\overline{m}}(X) = E_{\overline{m}} h$, but h is invariant and hence the two expectations are

equal (see, e.g., Lemma 6.3.1 of [55] or Lemma 7.5 of [58]). Thus we need only prove almost everywhere convergence in (4.1) to prove the theorem.

In this section we limit ourselves to the following special case of the theorem that can be proved using the ordinary ergodic theorem without any new techniques.

Lemma 4.1. *Given a finite-alphabet stationary kth order Markov source* $\{X_n\}$*, then there is an invariant function* h *such that*

$$\lim_{n\to\infty} \frac{-\ln m(X^n)}{n} = h;\ m - \text{a.e. } \textit{and in } L^1(m),$$

where h *is defined by*

$$h(x) = -E_{\overline{m}_x} \ln m(X_k|X^k), \tag{4.5}$$

where $\{\overline{m}_x\}$ *is the ergodic decomposition of the stationary mean* $\overline{m}$*. Furthermore,*

$$h(x) = \overline{H}_{\overline{m}_x}(X) = H_{\overline{m}_x}(X_k|X^k). \tag{4.6}$$

Proof of Lemma: We have that

$$-\frac{1}{n}\ln m(X^n) = -\frac{1}{n}\sum_{i=0}^{n-1} \ln m(X_i|X^i).$$

Since the process is kth order Markov with stationary transition probabilites, for $i > k$ we have that

$$m(X_i|X^i) = m(X_i|X_{i-k}, \cdots, X_{i-1}) = m(X_k|X^k)T^{i-k}.$$

The terms $-\ln m(X_i|X^i)$, $i = 0, 1, \cdots, k-1$ have finite expectation and hence are finite m-a.e. so that the ergodic theorem can be applied to deduce

$$\begin{aligned}
\frac{-\ln m(X^n)(x)}{n} &= -\frac{1}{n}\sum_{i=0}^{k-1} \ln m(X_i|X^i)(x) - \frac{1}{n}\sum_{i=k}^{n-1} \ln m(X_k|X^k)(T^{i-k}x) \\
&= -\frac{1}{n}\sum_{i=0}^{k-1} \ln m(X_i|X^i)(x) - \frac{1}{n}\sum_{i=0}^{n-k-1} \ln m(X_k|X^k)(T^i x) \\
&\underset{n\to\infty}{\to} E_{\overline{m}_x}(-\ln m(X_k|X^k)),
\end{aligned}$$

proving the first statement of the lemma. It follows from the ergodic decomposition of Markov sources (see Lemma 8.6.3 of [55] or Lemma 10.5 of [58]) that with probability 1, $\overline{m}_x(X_k|X^k) = m(X_k|\psi(x), X^k) = m(X_k|X^k)$, where ψ is the ergodic component function. This completes the proof. □

We prove the theorem in three steps: The first step considers stationary and ergodic sources and uses the approach of Ornstein and Weiss [141] (see also Shields [165]). The second step removes the requirement for ergodicity. This result will later be seen to provide an information theoretic interpretation of the ergodic decomposition. The third step extends the result to AMS processes by showing that such processes inherit limiting sample entropies from their stationary mean. The later extension of these results to more general relative entropy and information densities will closely parallel the proofs of the second and third steps for the finite case.

In subsequent chapters the definitions of entropy and information will be generalized and corresponding generalizations of the entropy ergodic theorem will be developed in Chapter 11.

4.2 Stationary Ergodic Sources

This section is devoted to proving the entropy ergodic theorem for the special case of stationary ergodic sources. The result was originally proved by Breiman [20]. The original proof first used the martingale convergence theorem to infer the convergence of conditional probabilities of the form $m(X_0|X_{-1}, X_{-2}, \cdots, X_{-k})$ to $m(X_0|X_{-1}, X_{-2}, \cdots)$. This result was combined with an an extended form of the ergodic theorem stating that if $g_k \to g$ as $k \to \infty$ and if g_k is L^1-dominated ($\sup_k |g_k|$ is in L^1), then $1/n \sum_{k=0}^{n-1} g_k T^k$ has the same limit as $1/n \sum_{k=0}^{n-1} g T^k$. Combining these facts yields that that

$$\frac{1}{n} \ln m(X^n) = \frac{1}{n} \sum_{k=0}^{n-1} \ln m(X_k|X^k) = \frac{1}{n} \sum_{k=0}^{n-1} \ln m(X_0|X_{-k}^k) T^k$$

has the same limit as

$$\frac{1}{n} \sum_{k=0}^{n-1} \ln m(X_0|X_{-1}, X_{-2}, \cdots) T^k$$

which, from the usual ergodic theorem, is the expectation

$$E(\ln m(X_0|\mathbf{X}^-) \equiv E(\ln m(X_0|X_{-1}, X_{-2}, \cdots)).$$

As suggested at the end of the preceeding chapter, this should be minus the conditional entropy $H(X_0|X_{-1}, X_{-2}, \cdots)$ which in turn should be the entropy rate $\overline{H}_X$. This approach has three shortcomings: it requires a result from martingale theory which has not been proved here or in the companion volume [55] or [58], it requires an extended ergodic theo-

rem which has similarly not been proved here, and it requires a more advanced definition of entropy which has not yet been introduced. Another approach is the sandwich proof of Algoet and Cover [7]. They show without using martingale theory or the extended ergodic theorem that $1/n \sum_{i=0}^{n-1} \ln m(X_0|X_{-i}^i)T^i$ is asymptotically sandwiched between the entropy rate of a kth order Markov approximation:

$$\frac{1}{n}\sum_{i=k}^{n-1} \ln m(X_0|X_{-k}^k)T^i \underset{n\to\infty}{\to} E_m[\ln m(X_0|X_{-k}^k)] = -H(X_0|X_{-k}^k)$$

and

$$\frac{1}{n}\sum_{i=k}^{n-1} \ln m(X_0|X_{-1}, X_{-2}, \cdots)T^i \underset{n\to\infty}{\to} E_m[\ln m(X_0|X_1, \cdots)] = -H(X_0|X_{-1}, X_{-2}, \cdots).$$

By showing that these two limits are arbitrarily close as $k \to \infty$, the result is proved. The drawback of this approach for present purposes is that again the more advanced notion of conditional entropy given the infinite past is required. Algoet and Cover's proof that the above two entropies are asymptotically close involves martingale theory, but this can be avoided by using Corollary 7.4 as will be seen.

The result can, however, be proved without martingale theory, the extended ergodic theorem, or advanced notions of entropy using the approach of Ornstein and Weiss [141], which is the approach we shall take in this chapter. In a later chapter when the entropy ergodic theorem is generalized to nonfinite alphabets and the convergence of entropy and information densities is proved, the sandwich approach will be used since the appropriate general definitions of entropy will have been developed and the necessary side results will have been proved.

Lemma 4.2. *Given a finite-alphabet source $\{X_n\}$ with a stationary ergodic distribution m, we have that*

$$\lim_{n\to\infty} \frac{-\ln m(X^n)}{n} = h;\ m - \text{a.e.},$$

where $h(x)$ is the invariant function defined by

$$h(x) = \overline{H}_m(X).$$

Proof: Define

$$h_n(x) = -\ln m(X^n)(x) = -\ln m(x^n)$$

and

$$\underline{h}(x) = \liminf_{n\to\infty} \frac{1}{n} h_n(x) = \liminf_{n\to\infty} \frac{-\ln m(x^n)}{n}.$$

Since $m((x_0, \cdots, x_{n-1})) \le m((x_1, \cdots, x_{n-1}))$, we have that

$$h_n(x) \ge h_{n-1}(Tx).$$

Dividing by n and taking the limit infimum of both sides shows that $\underline{h}(x) \ge \underline{h}(Tx)$. Since the $n^{-1}h_n$ are nonnegative and uniformly integrable (Lemma 3.7), we can use Fatou's lemma to deduce that $\underline{h}$ and hence also $\underline{h}T$ are integrable with respect to m. Integrating with respect to the stationary measure m yields

$$\int dm(x)\underline{h}(x) = \int dm(x)\underline{h}(Tx)$$

which can only be true if

$$\underline{h}(x) = \underline{h}(Tx); m - \text{a.e.},$$

that is, if $\underline{h}$ is an invariant function with m-probability one. If $\underline{h}$ is invariant almost everywhere, however, it must be a constant with probability one since m is ergodic (Lemma 6.7.1 of [55] or Lemma 7.12 of [58]). Since it has a finite integral (bounded by $\overline{H}_m(X)$), $\underline{h}$ must also be finite. Henceforth we consider $\underline{h}$ to be a finite constant.

We now proceed with steps that resemble those of the proof of the ergodic theorem in Section 7.2 of [55] or Section 8.1 or [58]. Fix $\epsilon > 0$. We also choose for later use a $\delta > 0$ small enough to have the following properties: If A is the alphabet of X_0 and $\|A\|$ is the finite cardinality of the alphabet, then

$$\delta \ln \|A\| < \epsilon, \tag{4.7}$$

and

$$-\delta \ln \delta - (1-\delta)\ln(1-\delta) \equiv h_2(\delta) < \epsilon. \tag{4.8}$$

The latter property is possible since $h_2(\delta) \to 0$ as $\delta \to 0$.

Define the random variable $n(x)$ to be the smallest integer n for which $n^{-1}h_n(x) \le \underline{h} + \epsilon$. As in the proof of the ergodic theorem, $n(x)$ in general will be large in order to well approximate the limit infimum, but by definition of the limit infimum there must be infinitely many n for which the inequality is true and hence $n(x)$ is everywhere finite, but it is not bounded. Still mimicking the proof of the ergodic theorem, define a set of "bad" sequences $B = \{x : n(x) > N\}$ where N is chosen large enough to ensure that $m(B) < \delta/2$. Define a bounded modification of $n(x)$ by

$$\tilde{n}(x) = \begin{cases} n(x) & x \notin B \\ 1 & x \in B \end{cases}$$

so that $\tilde{n}(x) \le N$ for all $x \in B^c$. We now parse the sequence into variable-length blocks. Iteratively define $n_k(x)$ by

$$\begin{aligned} n_0(x) &= 0 \\ n_1(x) &= \tilde{n}(x) \\ n_2(x) &= n_1(x) + \tilde{n}(T^{n_1(x)}x) = n_1(x) + l_1(x) \\ &\vdots \\ n_{k+1}(x) &= n_k(x) + \tilde{n}(T^{n_k(x)}x) = n_k(x) + l_k(x), \end{aligned}$$

where $l_k(x)$ is the length of the kth block:

$$l_k(x) = \tilde{n}(T^{n_k(x)}x).$$

We have parsed a very long sequence $x^L = (x_0, \cdots, x_{L-1})$, where $L \gg N$, into long blocks $x_{n_k(x)}, \cdots, x_{n_{k+1}(x)-1} = x_{n_k(x)}^{l_k(x)}$ which begin at time $n_k(x)$ and have length $l_k(x)$ for $k = 0, 1, \cdots$. We refer to this parsing as the *block decomposition* of a sequence. The kth block, which begins at time $n_k(x)$, must either have sample entropy satisfying

$$\frac{-\ln m(x_{n_k(x)}^{l_k(x)})}{l_k(x)} \le \underline{h} + \epsilon \tag{4.9}$$

or, equivalently, probability at least

$$m(x_{n_k(x)}^{l_k(x)}) \ge e^{-l_k(x)(\underline{h}+\epsilon)}, \tag{4.10}$$

or it must consist of only a single symbol. Blocks having length 1 ($l_k = 1$) could have the correct sample entropy, that is,

$$\frac{-\ln m(x_{n_k(x)}^{1})}{1} \le \overline{h} + \epsilon,$$

or they could be bad in the sense that they are the first symbol of a sequence with $n > N$; that is,

$$n(T^{n_k(x)}x) > N,$$

or, equivalently,

$$T^{n_k(x)}x \in B.$$

Except for these bad symbols, each of the blocks by construction will have a probability which satisfies the above bound.

Define for nonnegative integers n and positive integers l the sets

$$S(n, l) = \{x : m(X_n^l(x)) \ge e^{-l(\underline{h}+\epsilon)}\},$$

that is, the collection of infinite sequences for which (4.9) and (4.10) hold for a block starting at n and having length l. Observe that for such blocks there cannot be more than $e^{l(\underline{h}+\epsilon)}$ distinct l-tuples for which the

bound holds (lest the probabilities sum to something greater than 1). In symbols this is

$$||S(n,l)|| \le e^{l(\underline{h}+\epsilon)}. \tag{4.11}$$

The ergodic theorem will imply that there cannot be too many single symbol blocks with $n(T^{n_k(x)}x) > N$ because the event has small probability. These facts will be essential to the proof.

Even though we write $\tilde{n}(x)$ as a function of the entire infinite sequence, we can determine its value by observing only the prefix x^N of x since either there is an $n \le N$ for which $n^{-1}\ln m(x^n) \le \underline{h} + \epsilon$ or there is not. Hence there is a function $\hat{n}(x^N)$ such that $\tilde{n}(x) = \hat{n}(x^N)$. Define the finite length sequence event $C = \{x^N : \hat{n}(x^N) = 1 \text{ and } -\ln m(x^1) > \underline{h} + \epsilon\}$, that is, C is the collection of all N-tuples x^N that are prefixes of bad infinite sequences, sequences x for which $n(x) > N$. Thus in particular,

$$x \in B \text{ if and only if } x^N \in C. \tag{4.12}$$

Recall that we parse sequences of length $L \gg N$ and define the set G_L of "good" L-tuples by

$$G_L = \{x^L : \frac{1}{L-N}\sum_{i=0}^{L-N-1} 1_C(x_i^N) \le \delta\},$$

that is, G_L is the collection of all L-tuples which have fewer than $\delta(L-N) \le \delta L$ time slots i for which x_i^N is a prefix of a bad infinite sequence. From (4.12) and the ergodic theorem for stationary ergodic sources we know that m-a.e. we get an x for which

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} 1_C(x_i^N) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} 1_B(T^i x) = m(B) \le \frac{\delta}{2}. \tag{4.13}$$

From the definition of a limit, this means that with probability 1 we get an x for which there is an $L_0 = L_0(x)$ such that

$$\frac{1}{L-N}\sum_{i=0}^{L-N-1} 1_C(x_i^N) \le \delta; \text{ for all } L > L_0. \tag{4.14}$$

This follows because if the limit is less than $\delta/2$, there must be an L_0 so large that for larger L the time average is at least no greater than $2\delta/2 = \delta$. We can restate (4.14) as follows: with probability 1 we get an x for which $x^L \in G_L$ for all but a finite number of L. Stating this in negative fashion, we have one of the key properties required by the proof: If $x^L \in G_L$ for all but a finite number of L, then x^L cannot be in the complement G_L^c infinitely often, that is,

$$m(x : x^L \in G_L^c \text{ i.o.}) = 0. \tag{4.15}$$

We now change tack to develop another key result for the proof. For each L we bounded above the cardinality $||G_L||$ of the set of good L-tuples. By construction there are no more than δL bad symbols in an L-tuple in G_L and these can occur in any of at most

$$\sum_{k \le \delta L} \binom{L}{k} \le e^{h_2(\delta)L} \tag{4.16}$$

places, where we have used Lemma 3.6. Eq. (4.16) provides an upper bound on the number of ways that a sequence in G_L can be parsed by the given rules. The bad symbols and the final N symbols in the L-tuple can take on any of the $\|A\|$ different values in the alphabet. Eq. (4.11) bounds the number of finite length sequences that can occur in each of the remaining blocks and hence for any given block decomposition, the number of ways that the remaining blocks can be filled is bounded above by

$$\prod_{k:T^{n_k(x)}x \notin B} e^{l_k(x)(\underline{h}+\epsilon)} = e^{\sum_k l_k(x)(\underline{h}+\epsilon)} = e^{L(\underline{h}+\epsilon)}, \tag{4.17}$$

regardless of the details of the parsing. Combining these bounds we have that

$$||G_L|| \le e^{h_2(\delta)L} \times \|A\|^{\delta L} \times \|A\|^N \times e^{L(\underline{h}+\epsilon)} = e^{h_2(\delta)L + (\delta L + N)\ln\|A\| + L(\underline{h}+\epsilon)}$$

or

$$||G_L|| \le e^{L(\underline{h}+\epsilon+h_2(\delta)+(\delta+\frac{N}{L})\ln\|A\|)}.$$

Since δ satisfies (4.7)–(4.8), we can choose L_1 large enough so that $N \ln\|A\|/L_1 \le \epsilon$ and thereby obtain

$$||G_L|| \le e^{L(\underline{h}+4\epsilon)};\ L \ge L_1. \tag{4.18}$$

This bound provides the second key result in the proof of the lemma. We now combine (4.18) and (4.15) to complete the proof.

Let B_L denote a collection of L-tuples that are bad in the sense of having too large a sample entropy or, equivalently, too small a probability; that is if $x^L \in B_L$, then

$$m(x^L) \le e^{-L(\underline{h}+5\epsilon)}$$

or, equivalently, for any x with prefix x^L

$$h_L(x) \ge \underline{h} + 5\epsilon.$$

The upper bound on $||G_L||$ provides a bound on the probability of $B_L \bigcap G_L$:

$$m(B_L \bigcap G_L) = \sum_{x^L \in B_L \bigcap G_L} m(x^L) \le \sum_{x^L \in G_L} e^{-L(\underline{h}+5\epsilon)}$$

$$\le ||G_L|| e^{-L(\underline{h}+5\epsilon)} \le e^{-\epsilon L}.$$

Recall now that the above bound is true for a fixed $\epsilon > 0$ and for all $L \ge L_1$. Thus

$$\sum_{L=1}^{\infty} m(B_L \bigcap G_L) = \sum_{L=1}^{L_1-1} m(B_L \bigcap G_L) + \sum_{L=L_1}^{\infty} m(B_L \bigcap G_L)$$

$$\le L_1 + \sum_{L=L_1}^{\infty} e^{-\epsilon L} < \infty$$

and hence from the Borel-Cantelli lemma (Lemma 4.6.3 of [55] or Lemma 5.17 of [58]) $m(x : x^L \in B_L \bigcap G_L \text{ i.o.}) = 0$. We also have from (4.15), however, that $m(x : x^L \in G_L^c \text{ i.o. }) = 0$ and hence $x^L \in G_L$ for all but a finite number of L. Thus $x^L \in B_L$ i.o. if and only if $x^L \in B_L \bigcap G_L$ i.o. As this latter event has zero probability, we have shown that $m(x : x^L \in B_L$ i.o.) = 0 and hence

$$\limsup_{L \to \infty} h_L(x) \le \underline{h} + 5\epsilon.$$

Since ϵ is arbitrary we have proved that the limit supremum of the sample entropy $-n^{-1} \ln m(X^n)$ is less than or equal to the limit infimum and therefore that the limit exists and hence with m-probability 1

$$\lim_{n \to \infty} \frac{-\ln m(X^n)}{n} = \underline{h}. \tag{4.19}$$

Since the terms on the left in (4.19) are uniformly integrable from Lemma 3.7, we can integrate to the limit and apply Lemma 3.8 to find that

$$\underline{h} = \lim_{n \to \infty} \int dm(x) \frac{-\ln m(X^n(x))}{n} = \overline{H}_m(X),$$

which completes the proof of the lemma and hence also proves Theorem 4.1 for the special case of stationary ergodic measures. □

4.3 Stationary Nonergodic Sources

Next suppose that a source is stationary with ergodic decomposition $\{m_\lambda; \lambda \in \Lambda\}$ and ergodic component function ψ as in Theorem 1.6. The source will produce with probability one under m an ergodic component m_λ and Lemma 4.2 will hold for this ergodic component. In other words, we should have that

$$\lim_{n\to\infty} -\frac{1}{n}\ln m_\psi(X^n) = \overline{H}_{m_\psi}(X);\ m - \text{a.e.}, \tag{4.20}$$

that is,

$$m(\{x : -\lim_{n\to\infty} \ln m_{\psi(x)}(x^n) = \overline{H}_{m_{\psi(x)}}(X)\}) = 1.$$

This argument is made rigorous in the following lemma.

Lemma 4.3. *Suppose that $\{X_n\}$ is a stationary not necessarily ergodic source with ergodic component function ψ. Then*

$$m(\{x : -\lim_{n\to\infty} \ln m_{\psi(x)}(x^n) = \overline{H}_{m_{\psi(x)}}(X)\}) = 1;\ m - \text{a.e.}. \tag{4.21}$$

Proof: Let

$$G = \{x : -\lim_{n\to\infty} \ln m_{\psi(x)}(x^n) = \overline{H}_{m_{\psi(x)}}(X)\}$$

and let G_λ denote the section of G at λ, that is,

$$G_\lambda = \{x : -\lim_{n\to\infty} \ln m_\lambda(x^n) = \overline{H}_{m_\lambda}(X)\}.$$

From the ergodic decomposition (e.g., Theorem 1.6 or [55], Theorem 8.5.1, [58], Theorem 10.1) and (1.28)

$$m(G) = \int dP_\psi(\lambda) m_\lambda(G),$$

where

$$m_\lambda(G) = m(G|\psi = \lambda) = m(G \bigcap \{x : \psi(x) = \lambda\}|\psi = \lambda)$$

$$= m(G_\lambda|\psi = \lambda) = m_\lambda(G_\lambda)$$

which is 1 for all λ from the stationary ergodic result. Thus

$$m(G) = \int dP_\psi(\lambda) m_\lambda(G_\lambda) = 1.$$

It is straightforward to verify that all of the sets considered are in fact measurable. □

Unfortunately it is not the sample entropy using the distribution of the ergodic component that is of interest, rather it is the original sample entropy for which we wish to prove convergence. The following lemma shows that the two sample entropies converge to the same limit and hence Lemma 4.3 will also provide the limit of the sample entropy with respect to the stationary measure.

Lemma 4.4. *Given a stationary source $\{X_n\}$, let $\{m_\lambda;\ \lambda \in \Lambda\}$ denote the ergodic decomposition and ψ the ergodic component function of Theorem 1.6. Then*

$$\lim_{n\to\infty} \frac{1}{n} \ln \frac{m_\psi(X^n)}{m(X^n)} = 0;\ m - \text{a.e.}$$

Proof: First observe that if $m(a^n)$ is 0, then from the ergodic decomposition with probability 1 $m_\psi(a^n)$ will also be 0. One part is easy. For any $\epsilon > 0$ we have from the Markov inequality that

$$m(\frac{1}{n} \ln \frac{m(X^n)}{m_\psi(X^n)} > \epsilon) = m(\frac{m(X^n)}{m_\psi(X^n)} > e^{n\epsilon}) \le E_m(\frac{m(X^n)}{m_\psi(X^n)}) e^{-n\epsilon}.$$

The expectation, however, can be evaluated as follows: Let $A_n^{(\lambda)} = \{a^n : m_\lambda(a^n) > 0\}$. Then

$$E_m\left(\frac{m(X^n)}{m_\psi(X^n)}\right) = \int dP_\psi(\lambda) \sum_{a^n \in A_n} \frac{m(a^n)}{m_\lambda(a^n)} m_\lambda(a^n) = \int dP_\psi(\lambda) m(A_n^{(\lambda)}) \le 1,$$

where P_ψ is the distribution of ψ. Thus

$$m(\frac{1}{n} \ln \frac{m(X^n)}{m_\psi(X^n)} > \epsilon) \le e^{-n\epsilon}.$$

and hence

$$\sum_{n=1}^{\infty} m(\frac{1}{n} \ln \frac{m(X^n)}{m_\psi(X^n)} > \epsilon) < \infty$$

and hence from the Borel-Cantelli lemma

$$m(\frac{1}{n} \ln \frac{m(X^n)}{m_\psi(X^n)} > \epsilon \text{ i.o.}) = 0$$

and hence with m probability 1

$$\limsup_{n\to\infty} \frac{1}{n} \ln \frac{m(X^n)}{m_\psi(X^n)} \le \epsilon.$$

Since ϵ is arbitrary,

$$\limsup_{n\to\infty} \frac{1}{n} \ln \frac{m(X^n)}{m_\psi(X^n)} \le 0;\ m - \text{a.e.} \tag{4.22}$$

For later use we restate this as

$$\liminf_{n\to\infty} \frac{1}{n} \ln \frac{m_\psi(X^n)}{m(X^n)} \ge 0;\ m - \text{a.e.} \tag{4.23}$$

Now turn to the converse inequality. For any positive integer k, we can construct a stationary k-step Markov approximation to m as in Section 3.7 that is, construct a process $m^{(k)}$ with the conditional probabilities

$$m^{(k)}(X_n \in F|X^n) = m^{(k)}(X_n \in F|X_{n-k}^k) = m(X_n \in F|X_{n-k}^k)$$

and the same kth order distributions $m^{(k)}(X^k \in F) = m(X^k \in F)$. Consider the probability

$$m(\frac{1}{n}\ln\frac{m^{(k)}(X^n)}{m(X^n)} \geq \epsilon) = m(\frac{m^{(k)}(X^n)}{m(X^n)} \geq e^{n\epsilon}) \leq E_m(\frac{m^{(k)}(X^n)}{m(X^n)})e^{-n\epsilon}.$$

The expectation is evaluated as

$$\sum_{x^n}\frac{m^{(k)}(x^n)}{m(x^n)}m(x^n) = 1$$

and hence we again have using Borel-Cantelli that

$$\limsup_{n\to\infty}\frac{1}{n}\ln\frac{m^{(k)}(X^n)}{m(X^n)} \leq 0.$$

Apply the usual ergodic theorem to conclude that with probability 1 under m

$$\limsup_{n\to\infty}\frac{1}{n}\ln\frac{1}{m(X^n)} \leq \lim_{n\to\infty}\frac{1}{n}\ln\frac{1}{m^{(k)}(X^n)} = E_{m_\psi}[-\ln m(X_k|X^k)].$$

Combining this result with (4.20) and Lemma 3.10 yields

$$\limsup_{n\to\infty}\frac{1}{n}\ln\frac{m_\psi(X^n)}{m(X^n)} \leq -\overline{H}_{m_\psi}(X) - E_{m_\psi}[\ln m(X_k|X^k)]. = \overline{H}_{m_\psi||m^{(k)}}(X).$$

This bound holds for any integer k and hence it must also be true that m-a.e. the following holds:

$$\limsup_{n\to\infty}\frac{1}{n}\ln\frac{m_\psi(X^n)}{m(X^n)} \leq \inf_k \overline{H}_{m_\psi||m^{(k)}}(X) \equiv \zeta. \tag{4.24}$$

In order to evaluate ζ we apply the ergodic decomposition of relative entropy rate (Corollary 3.5) and the ordinary ergodic decomposition to write

$$\begin{aligned}\int dP_\psi \zeta &= \int dP_\psi \inf_k \overline{H}_{m_\psi||m^{(k)}}(X)\\ &\leq \inf_k \int dP_\psi \overline{H}_{m_\psi||m^{(k)}}(X) = \inf_k \overline{H}_{m||m^{(k)}}(X).\end{aligned}$$

From Theorem 3.4, the right hand term is 0. If the integral of a nonnegative function is 0, the integrand must itself be 0 with probability one. Thus (4.24) becomes

$$\limsup_{n\to\infty} \frac{1}{n} \ln \frac{m_\psi(X^n)}{m(X^n)} \le 0,$$

which with (4.23) completes the proof of the lemma. □

We shall later see that the quantity

$$i_n(X^n;\psi) = \frac{1}{n} \ln \frac{m_\psi(X^n)}{m(X^n)}$$

is the sample mutual information (in a generalized sense so that it applies to the usually non-discrete ψ) and hence the lemma states that the normalized sample mutual information between the process outputs and the ergodic component function goes to 0 as the number of samples goes to infinity.

The two previous lemmas immediately yield the following result.

Corollary 4.1. *The conclusions of Theorem 4.1 hold for sources that are stationary.*

4.4 AMS Sources

The principal idea required to extend the entropy theorem from stationary sources to AMS sources is contained in Lemma 4.6. It shows that an AMS source inherits sample entropy properties from an asymptotically dominating stationary source (just as it inherits ordinary ergodic properties from such a source). The result is originally due to Gray and Kieffer [62], but the proof here is somewhat different. The tough part here is handling the fact that the sample average being considered depends on a specific measure. From Theorem 1.2, the stationary mean of an AMS source dominates the original source on tail events, that is, events in $\mathcal{F}_\infty$. We begin by showing that certain important events can be recast as tail events, that is, they can be determined by looking at only samples in the arbitrarily distant future. The following result is of this variety: It implies that sample entropy is unaffected by the starting time.

Lemma 4.5. *Let $\{X_n\}$ be a finite-alphabet source with distribution m. Recall that $X_k^n = (X_k, X_{k+1}, \cdots, X_{k+n-1})$ and define the information density*

$$i(X^k; X_k^{n-k}) = \ln \frac{m(X^n)}{m(X^k)m(X_k^{n-k})}.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} i(X^k; X_k^{n-k}) = 0; \ m - \text{a.e.}$$

Comment: The lemma states that with probability 1 the per-sample mutual information density between the first k samples and future samples goes to zero in the limit. Equivalently, limits of $n^{-1} \ln m(X^n)$ will be the same as limits of $n^{-1} \ln m(X_k^{n-k})$ for any finite k. Note that the result does not require even that the source be AMS. The lemma is a direct consequence of Lemma 3.19.
Proof: Define the distribution $p = m_{X^k} \times m_{X_k, X_{k+1}, \dots}$, that is, a distribution for which all samples after the first k are independent of the first k samples. Thus, in particular, $p(X^n) = m(X^k) m(X_k^n)$. We will show that $p \gg m$, in which case the lemma will follow from Lemma 3.19. Suppose that $p(F) = 0$. If we denote $X_k^+ = X_k, X_{k+1}, \cdots$, then

$$0 = p(F) = \sum_{x^k} m(x^k) m_{X_k^+}(F_{x^k}),$$

where F_{x^k} is the section $\{x_k^+ : (x^k, x_k^+) = x \in F\}$. For the above relation to hold, we must have $m_{X_k^+}(F_{x^k}) = 0$ for all x^k with $m(x^k) \neq 0$. We also have, however, that

$$m(F) = \sum_{a^k} m(X^k = a^k, X_k^+ \in F_{a^k})$$

$$= \sum_{a^k} m(X^k = a^k | X_k^+ \in F_{a^k}) m(X_k^+ \in F_{a^k}).$$

But this sum must be 0 since the rightmost terms are 0 for all a^k for which $m(X^k = a^k)$ is not 0. (Observe that we must have $m(X^k = a^k | X_k^+ \in F_{a^k}) = 0$ if $m(X_k^+ \in F_{a^k}) \neq 0$ since otherwise $m(X^k = a^k) \geq m(X^k = a^k, X_k^+ \in F_{a^k}) > 0$, yielding a contradiction.) Thus $p \gg m$ and the lemma is proved. □

For later use we note that we have shown that a joint distribution is dominated by a product of its marginals if one of the marginal distributions is discrete.

Lemma 4.6. *Suppose that $\{X_n\}$ is an AMS source with distribution m and suppose that $\overline{m}$ is a stationary source that asymptotically dominates m (e.g., $\overline{m}$ is the stationary mean). If there is an invariant function h such that*

$$\lim_{n \to \infty} -\frac{1}{n} \ln \overline{m}(X^n) = h; \ \overline{m} - \text{a.e.},$$

then also,

$$\lim_{n \to \infty} -\frac{1}{n} \ln m(X^n) = h; \ m - \text{a.e.}$$

Proof: For any k we can write using the chain rule for densities

$$-\frac{1}{n}\ln m(X^n) + \frac{1}{n}\ln m(X_k^{n-k}) = -\frac{1}{n}\ln m(X^k|X_k^{n-k})$$
$$= -\frac{1}{n}i(X^k;X_k^{n-k}) - \frac{1}{n}\ln m(X^k).$$

From the previous lemma and from the fact that $H_m(X^k) = -E_m \ln m(X^k)$ is finite, the right hand terms converge to 0 as $n \to \infty$ and hence for any k

$$\lim_{n\to\infty} -\frac{1}{n}\ln m(X^k|X_k^{n-k}) =$$
$$\lim_{n\to\infty}(-\frac{1}{n}\ln m(X^n) + \frac{1}{n}\ln m(X_k^{n-k})) = 0;\ m-\text{a.e.} \quad (4.25)$$

This implies that there is a subsequence $k(n) \to \infty$ such that

$$-\frac{1}{n}\ln m(X^{k(n)}|X_{k(n)}^{n-k(n)}) = -\frac{1}{n}\ln m(X^n) - \frac{1}{n}\ln m(X_{k(n)}^{n-k}(n)) \to 0;\ m-\text{a.e.} \quad (4.26)$$

To see this, observe that (4.25) ensures that for each k there is an $N(k)$ large enough so that $N(k) > N(k-1)$ and

$$m(|-\frac{1}{N(k)}\ln m(X^k|X_k^{N(k)-k})| > 2^{-k}) \le 2^{-k}. \quad (4.27)$$

Applying the Borel-Cantelli lemma implies that for any ϵ,

$$m(|-1/N(k)\ln m(X^k|X_k^{N(k)-k})| > \epsilon \text{ i.o.}) = 0.$$

Now let $k(n) = k$ for $N(k) \le n < N(k+1)$. Then

$$m(|-1/n\ln m(X^{k(n)}|X_{k(n)}^{n-k(n)})| > \epsilon \text{ i.o.}) = 0$$

and therefore

$$\lim_{n\to\infty}\left(-\frac{1}{n}\ln m(X^n) + \frac{1}{n}\ln m(X_{k(n)}^{n-k(n)})\right) = 0;\ m-\text{a.e.}$$

as claimed in (4.26).

In a similar manner we can also choose the sequence so that

$$\lim_{n\to\infty}\left(-\frac{1}{n}\ln \overline{m}(X^n) + \frac{1}{n}\ln \overline{m}(X_{k(n)}^{n-k(n)})\right) = 0;\ \overline{m}-\text{a.e.},$$

that is, we can choose $N(k)$ so that (4.27) simultaneously holds for both m and $\overline{m}$. Invoking the entropy ergodic theorem for the stationary $\overline{m}$ (Corollary 4.3) we have therefore that

$$\lim_{n\to\infty} -\frac{1}{n}\ln \overline{m}(X_{k(n)}^{n-k(n)}) = \overline{h};\ \overline{m}-\text{a.e.}. \quad (4.28)$$

From Markov's inequality (Lemma 4.4.3 of [55] or Lemma 5.8 of [58])

$$\begin{aligned}\overline{m}(-\frac{1}{n}\ln m(X_k^n) \le -\frac{1}{n}\ln \overline{m}(X_k^n) - \epsilon) &= \overline{m}(\frac{m(X_k^n)}{\overline{m}(X_k^n)} \ge e^{n\epsilon}) \\ &\le e^{-n\epsilon} E_{\overline{m}} \frac{m(X_k^{n-k})}{\overline{m}(X_k^{n-k})} \\ &= e^{-n\epsilon} \sum_{x_k^{n-k}:\overline{m}(x_k^{n-k})\neq 0} \frac{m(x_k^{n-k})}{\overline{m}(x_k^{n-k})} \overline{m}(x_k^{n-k}) \\ &\le e^{-n\epsilon}.\end{aligned}$$

Hence taking $k = k(n)$ and again invoking the Borel-Cantelli lemma we have that

$$\overline{m}(-\frac{1}{n}\ln m(X_{k(n)}^{n-k(n)}) \le -\frac{1}{n}\ln \overline{m}(X_{k(n)}^{n-k(n)}) - \epsilon \text{ i.o.}) = 0$$

or, equivalently, that

$$\liminf_{n\to\infty} -\frac{1}{n}\ln \frac{m(X_{k(n)}^{n-k(n)})}{\overline{m}(X_{k(n)}^{n-k(n)})} \ge 0;\ \overline{m}\text{ – a.e.} \tag{4.29}$$

Therefore from (4.28)

$$\liminf_{n\to\infty} -\frac{1}{n}\ln m(X_{k(n)}^{n-k(n)}) \ge h;\ \overline{m}\text{ – a.e..} \tag{4.30}$$

The above event is in the tail σ-field $\mathcal{F}_\infty = \bigcap_n \sigma(X_n, X_{n+1}, \cdots)$ since it can be determined from $X_{k(n)}, \cdots$ for arbitrarily large n and since h is invariant. Since $\overline{m}$ dominates m on the tail σ-field (Theorem 1.3), we have also

$$\liminf_{n\to\infty} -\frac{1}{n}\ln m(X_{k(n)}^{n-k(n)}) \ge h;\ m\text{ – a.e.}$$

and hence by (4.26)

$$\liminf_{n\to\infty} -\frac{1}{n}\ln m(X^n) \ge h;\ m\text{ – a.e.}$$

which proves half of the lemma. Since

$$\overline{m}(\lim_{n\to\infty} -\frac{1}{n}\ln \overline{m}(X^n) \ne h) = 0$$

and since $\overline{m}$ asymptotically dominates m (Theorem 1.2), given $\epsilon > 0$ there is a k such that

$$m(\lim_{n\to\infty} -\frac{1}{n}\ln \overline{m}(X_k^n) = h) \ge 1 - \epsilon.$$

Again applying Markov's inequality and the Borel-Cantelli lemma as in the development of (4.28) we have that

$$\liminf_{n\to\infty} -\frac{1}{n}\ln\frac{\overline{m}(X_k^n)}{m(X_k^n)} \geq 0;\ m-\text{a.e},$$

which implies that

$$m(\limsup_{n\to\infty} -\frac{1}{n}\ln m(X_k^n) \leq h) \geq 1-\epsilon$$

and hence also that

$$m(\limsup_{n\to\infty} -\frac{1}{n}\ln m(X^n) \leq h) \geq 1-\epsilon.$$

Since ϵ can be made arbitrarily small, this proves that m-a.e.

$$\limsup_{n\to\infty} -n^{-1}\ln m(X^n) \leq h,$$

which completes the proof of the lemma. □

The lemma combined with Corollary 4.3 completes the proof of Theorem 4.1. □

Theorem 4.1 and Lemma 2.5 immediately yield the following corollary stating that a stationary coding of an AMS process has a well defined entropy rate given by a limit, as in the case of a stationary process.

Corollary 4.2. *Theorem 4.1 If f is a stationary coding of an AMS process, then*

$$\overline{H}(f) = \lim_{n\to\infty}\frac{1}{n}H(f^n).$$

4.5 The Asymptotic Equipartition Property

Since convergence almost everywhere implies convergence in probability, Theorem 4.1 has the following implication: Suppose that $\{X_n\}$ is an AMS ergodic source with entropy rate $\overline{H}$. Given $\epsilon > 0$ there is an N such that for all $n > N$ the set

$$G_n = \{x^n : |n^{-1}h_n(x) - \overline{H}| \geq \epsilon\} = \{x^n : e^{-n(\overline{H}+\epsilon)} \leq m(x^n) \leq e^{-n(\overline{H}-\epsilon)}\}$$

has probability greater then $1-\epsilon$. Furthermore, as in the proof of the theorem, there can be no more than $e^{n(\overline{H}+\epsilon)}$ n-tuples in G_n. Thus there are two sets of n-tuples: a "good" set of approximately $e^{n\overline{H}}$ n-tuples having approximately equal probability of $e^{-n\overline{H}}$ and the complement of this set which has small total probability. The set of good sequences are

often referred to as "typical sequences" or "entropy-typical sequences" in the information theory literature and in this form the theorem is called the asymptotic equipartition property or the AEP.

As a first information theoretic application of an ergodic theorem, we consider a simple coding scheme called an "almost noiseless" or "almost lossless" source code. As we often do, we consider logarithms to the base 2 when considering specific coding applications. Suppose that a random process $\{X_n\}$ has a finite alphabet A with cardinality $\|A\|$ and entropy rate $\overline{H}$. Suppose that $\overline{H} < \log \|A\|$, e.g., A might have 16 symbols, but the entropy rate is slightly less than 2 bits per symbol rather than $\log 16 = 4$. Larger alphabets cost money in either storage or communication applications. For example, to communicate a source with a 16 letter alphabet sending one letter per second without using any coding and using a binary communication system we would need to send 4 binary symbols (or four *bits*) for each source letter and hence 4 bits per second would be required. If the alphabet only had 4 letters, we would need to send only 2 bits per second. The question is the following: Since our source has an alphabet of size 16 but an entropy rate of less than 2, can we code the original source into a new source with an alphabet of only $4 = 2^2$ letters so as to communicate the source at the smaller rate and yet have the receiver be able to recover the original source? The AEP suggests a technique for accomplishing this provided we are willing to tolerate rare errors.

We construct a block code of the original source by first picking a small ϵ and a δ small enough so that $\overline{H} + \delta < 2$. Choose a large enough n so that the AEP holds giving a set G_n of good sequences as above with probability greater than $1 - \epsilon$. Index this collection of fewer than $2^{n(\overline{H}+\delta)} < 2^{2n}$ sequences using binary $2n$-tuples. The source X_k is parsed into blocks of length n as $X_{kn}^n = (X_{kn}, X_{kn+1}, \cdots, X_{(k+1)n})$ and each block is encoded into a binary $2n$-tuple as follows: If the source n-tuple is in G_n, the codeword is its binary $2n$-tuple index. Select one of the unused binary $2n$-tuples as the error index and whenever an n-tuple is not in G_n, the error index is the codeword. The receiver or decoder than uses the received index and decodes it as the appropriate n-tuple in G_n. If the error index is received, the decoder can declare an arbitrary source sequence or just declare an error. With probability at least $1 - \epsilon$ a source n-tuple at a particular time will be in G_n and hence it will be correctly decoded. We can make this probability as small as desired by taking n large enough, but we cannot in general make it 0.

The above simple scheme is an example of a block coding scheme as considered in Section 2.7. If considered as a mapping from sequences into sequences, the map is not stationary, but it is block stationary in the sense that shifting an input block by n results in a corresponding block shift of the encoded sequence by $2n$ binary symbols.

Chapter 5
Distortion and Approximation

Abstract Various notions of the distortion between random variables, vectors, and processes as well as between different codings of a common source are quantified in this chapter. A distortion measure is not a "measure" in the sense used so far — it is an assignment of a nonnegative real number which indicates how bad an approximation one symbol or random object or coding is of another. The smaller the distortion, the better the approximation. If the two objects correspond to the input and output of a communication system, then the average distortion provides a measure of the *performance* or *fidelity* of the system. Small average distortion means high fidelity and good performance, while large average distortion means low fidelity and poor performance. Distortion measures generalize the idea of a distance or metric and they need not have metric properties such as the triangle inequality and symmetry, but such properties can be exploited when available and unsurprisingly the most important notions of distortion are either metrics or simple functions of metrics. We shall encounter several notions of distortion and a diversity of applications, with the most important application being the average distortion between input and output as a measure of the performance of a communications system. Other applications include extensions of finite memory channels to channels which approximate finite memory channels, geometric characterizations of the optimal performance of communications systems, approximations of complicated codes by simpler ones, and modeling random processes.

5.1 Distortion Measures

Given two measurable spaces $(A, \mathcal{B}_A)$ and $(B, \mathcal{B}_B)$, a *distortion measure* on $A \times B$ is a nonnegative measurable mapping $\rho : A \times B \to [0, \infty)$ which assigns a real number $\rho(x, y)$ to each $x \in A$ and $y \in B$ which can be

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thought of as the cost of reproducing x and y. The principal practical goal is to have a number by which the goodness or badness of communication systems can be compared. For example, if the input to a communication system is a random variable $X \in A$ and the output is $Y \in B$, then one possible measure of the performance or quality of the system is the average distortion $E\rho(X, Y)$. A distortion measure is essentially the same as a loss, risk, or cost function in statistics.

Ideally one would like a distortion measure to have three properties:

- It should be tractable so that one can do useful theory.
- It should be computable so that it can be measured in real systems.
- It should be subjectively meaningful in the sense that small (large) distortion corresponds to good (bad) perceived quality.

Unfortunately these requirements are often incompatible and one is forced to compromise between tractability and subjective significance in the choice of distortion measures. Among the most popular choices for distortion measures are metrics or distances, but some practically important distortion measures are not metrics in that they are not symmetric or do not satisfy the triangle inequality.

Two specific examples are by far the most important in information theory, signal processing, communications, and statistics: the per symbol Hamming distortion and squared-error distortion. While neither provides a panacea, the Hamming distortion is the most common distortion measure for discrete alphabets and the squared-error for continuous alphabets for several reasons. Both are tractable and simple and easy to compute. The Hamming distortion is arguably the most unforgiving distortion measure possible since the maximum distortion is assigned to every pair unless they two match exactly. Average closeness with respect to any distortion measure with a maximum value can be assured by ensuring average closeness in the Hamming sense. The Hamming distance is primarily used with discrete alphabet variables. Squared-error does not play a similar "worst case" role, but it is an intuitive measure since its average is the energy of the error between two variables. Variations allowing linear weightings of variables and signals before evaluating the average of a quadratic error yield a variety of distortion measures that have proved useful in speech, voice, audio, image, and video processing, especially in techniques incorporating "perceptual coding" where distortion is measured according to its estimated impact on human perception.

Suppose that $A, B \subset \mathbb{R}$. The *Hamming distance* is defined by

$$\rho(x, y) = d_H(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

The Hamming distance is a distortion measure which is also a metric. Given a probability measure p on $(A \times B, \mathcal{B}(A \times B))$ let X and Y denote

the coordinate random variables $X(x, y) = x$, $Y(x, y) = y$, then

$$E_p d_H = \Pr(X \neq Y).$$

The *squared-error distortion* is defined by

$$\rho(x, y) = | x - y |^2 .$$

Squared error is not a metric, but it is the square of the metric $| x - y |$, which means it inherits properties of a metric: it is symmetric and its square root satisfies the triangle inequality. The average squared error $E[(X - Y)^2]$ is often interpreted as the energy in the error signal.

The Hamming distance is well-defined if we consider vector alphabets $A, B \subset \mathbb{R}^n$: $d_{\mathrm{H}}(x^n, y^n) = 0$ if $x^n = y^n$ and 0 otherwise. A far more useful extension from scalars to vectors, however, is construction of the vector distortion from the scalar Hamming distance in an additive or linear fashion:

$$\rho(x^n, y^n) = \sum_{i=0}^{n-1} d(x_i, y_i), \tag{5.1}$$

which is the number of coordinates in which the two vectors differ. This distortion measure is referred to as the *average Hamming distance* or *mean Hamming distance*.

Extending the squared-error distortion in a similar additive fashion to vector spaces yields a squared-error distortion

$$\rho(x^n, y^n) = \sum_{i=0}^{n-1} | x_i - y_i |^2 . \tag{5.2}$$

This is not a metric, but it is the square of the Euclidean or ℓ_2 distance:

$$\rho(x^n, y^n) = \|x^n - y_n\|_2^2.$$

These examples typify distortion measures and are the most important special cases, but most of the results hold more generally and the development will focus on distortion measures formed as a positive power of a metric. Many of the results developed here will be for the case where A is a Polish space, a complete separable metric space under a metric d, and B is either A itself or a Borel subset of A. The distortion measure is assumed to be a positive power of d. In this case the distortion measure is fundamental to the structure of the alphabet and the alphabets are standard since the space is Polish.

5.2 Fidelity Criteria

It is often of interest to consider distortion between sequences as well as between scalars and vectors. One approach to a distortion between sequences is to define a family of distortion measures between vectors of all dimensions and consider limits. This is the idea behind a fidelity criterion as defined by Shannon [162, 163]. Given "scalar" spaces A and B, a *fidelity criterion* ρ_n, $n = 1, 2, \cdots$, is a sequence of distortion measures on $A^n \times B^n$. A candidate for the distortion between infinite sequences is then the limit supremum of the per symbol distortion

$$\rho_\infty(x, y) = \limsup_{n\to\infty} \frac{1}{n}\rho_n(x^n, y^n).$$

If one has a pair random process, say $\{X_n, Y_n\}$ with process distribution p, then it is of interest to find conditions under which there is a limiting per symbol distortion in the sense that the limit exists with p probability 1:

$$\rho_\infty(x, y) = \lim_{n\to\infty} \frac{1}{n}\rho_n(x^n, y^n). \tag{5.3}$$

As one might guess, the distortion measures in the sequence need to be interrelated in order to get useful behavior. The simplest and most common example is that of an *additive* or *single-letter* fidelity criterion which has the form

$$\rho_n(x^n, y^n) = \sum_{i=0}^{n-1} \rho_1(x_i, y_i).$$

Here if the pair process is AMS and ρ_1 satisfies suitable integrability assumptions, then the limiting per-symbol distortion

$$\frac{1}{n}\rho_n(x^n, y^n) = \frac{1}{n}\sum_{i=0}^{n-1} \rho_1(x_i, y_i).$$

will exist and be invariant from the ergodic theorem.

If the pair process is stationary rather than only AMS, then one can consider the more general case where ρ_n is subadditive in the sense that

$$\rho_n(x^n, y^n) \le \rho_k(x^k, y^k) + \rho_{n-k}(x_k^{n-k}, y_k^{n-k}).$$

In this case then stationarity of the pair process and integrability of ρ_1 will ensure that $n^{-1}\rho_n$ converges from the subadditive ergodic theorem. For example, if d is a distortion measure (possibly a metric) on $A \times B$, then

$$\rho_n(x^n, y^n) = \left(\sum_{i=0}^{n-1} d(x_i, y_i)^p \right)^{1/p}$$

for $p > 1$ is subadditive from Minkowski's inequality.

By far the bulk of the information theory literature considers only additive fidelity criteria and we will share this emphasis. The most common examples of additive fidelity criteria involve defining the per-letter distortion ρ_1 in terms of an underlying metric d. For example, set $\rho_1(x, y) = d(x, y)$, in which case ρ_n is also a metric for all n, or $\rho_1(x, y) = d^p(x, y)$ for some $p > 0$. If $1 > p > 0$, then again ρ_n is a metric for all n. If $p \geq 1$, then ρ_n is not a metric, but $\rho_n^{1/p}$ is a metric. We do not wish to include the $1/p$ in the definition of the fidelity criterion, however, because what we gain by having a metric distortion we more than lose when we take the expectation. For example, a popular distortion measure is the expected squared error, not the expectation of the square root of the squared error.

The fidelity criteria introduced here all are *context-free* in that the distortion between n successive input/output samples of a pair process does not depend on samples occurring before or after these n-samples. Some work has been done on context-dependent distortion measures (see, e.g., [107]), but we do not consider their importance sufficient to merit the increased notational and technical difficulties involved. Hence we shall consider only context-free distortion measures.

5.3 Average Limiting Distortion

Suppose that $\{X_n, Y_n\}$ is an AMS pair process with alphabet $A \times B$. Let p denote the corresponding distribution of the pair process. One measure of the quality (or rather the lack thereof) of approximation of X by Y is given by the average limiting distortion with respect to a fidelity criterion. Given two sequences x and y and a fidelity criterion ρ_n; $n = 1, 2, \cdots$, define the limiting sample average distortion or *sequence distortion* by

$$\rho_\infty(x, y) = \limsup_{n \to \infty} \frac{1}{n} \rho_n(x^n, y^n)$$

and define the average sequence distortion

$$\Delta(p) = E_p \rho_\infty = E_p \left(\limsup_{n \to \infty} \frac{1}{n} \rho_n(X^n, Y^n) \right).$$

We focus on two important special cases. The first and most important is that of AMS pair processes and additive fidelity criteria. We also consider the case of subadditive distortion measures and systems that are either two-sided and AMS or are one-sided and stationary. Unhappily the overall AMS one-sided case cannot be handled as there is not yet a general subadditive ergodic theorem for that case (from Theorem 8.5 of [58] there is a theorem if in addition p is absolutely continuous with respect to its stationary mean $\overline{p}$). In all of these cases we have that if ρ_1 is integrable with respect to the stationary mean process $\overline{p}$, then

$$\rho_\infty(x, y) = \lim_{n\to\infty} \frac{1}{n}\rho_n(x^n, y^n);\ p - \text{a.e.}, \tag{5.4}$$

and ρ_∞ is an invariant function of its two arguments, i.e.,

$$\rho_\infty(T_A x, T_B y) = \rho_\infty(x, y);\ p - \text{a.e.}. \tag{5.5}$$

When a process distribution and fidelity criterion are such that (5.4) and (5.5) are satisfied (at least with probability 1) we say that we have a *convergent fidelity criterion*. This property holds, for example, by an underlying assumption that p is AMS and that the fidelity criterion is additive and the single-letter distortion is integrable with respect to the stationary mean.

Since ρ_∞ is invariant, we have from Lemma 6.3.1 of [55] or Corollary 7.10 of [58] that

$$\Delta(p) = E_p\rho_\infty = E_{\overline{p}}\rho_\infty. \tag{5.6}$$

If the fidelity criterion is additive, then we have from the stationarity of $\overline{p}$ that the average limiting distortion is given by

$$\Delta(p) = E_{\overline{p}}\rho_1(X_0, Y_0). \tag{5.7}$$

If the fidelity criterion is subadditive and the processes stationary, then this is replaced by

$$\Delta(p) = \inf_N \frac{1}{N} E_p\rho_N(X^N, Y^N). \tag{5.8}$$

Assume for the remainder of this section that ρ_n is an additive fidelity criterion. Suppose that we know that p is N-stationary; that is, if $T = T_A \times T_B$ denotes the shift on the input/output space $A^{\mathbb{T}} \times B^{\mathbb{T}}$, then the overall process is stationary with respect to T^N. In this case

$$\Delta(p) = \frac{1}{N} E_p\rho_N(X^N, Y^N). \tag{5.9}$$

We can also consider the behavior of the N-shift more generally when the system is only AMS This will be useful when considering block codes.

Suppose now that p is AMS with stationary mean $\overline{p}$. Then from Theorem 7.3.1 of [55] or Theorem 8.2 of [58], p is also T^N-AMS with an N-stationary mean, say $\overline{p}_N$. Applying the ergodic theorem to the N shift then implies that if ρ_N is $\overline{p}_N$-integrable, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \rho_N(x_{iN}^N, \hat{x}_{iN}^N) = \rho_\infty^{(N)} \tag{5.10}$$

exists $\overline{p}_N$ (and hence also p) almost everywhere. In addition, $\rho_\infty^{(N)}$ is N-invariant and

$$E_p \rho_\infty^{(N)} = E_{\overline{p}_N} \rho_\infty^{(N)} = E_{\overline{p}_N} \rho_N(X^N, \hat{X}^N). \tag{5.11}$$

Comparison of (5.4) and (5.11) shows that $\rho_\infty^{(N)} = N\rho_\infty$ p-a.e. and hence

$$\Delta(p) = \frac{1}{N} E_{\overline{p}_N} \rho_N(X^N, \hat{X}^N) = \frac{1}{N} E_p \rho_\infty^{(N)} = E_{\overline{p}} \rho_1(X_0, \hat{X}_0) = \Delta(\overline{p}). \tag{5.12}$$

The key point here is that the measure of the quality or fidelity in terms of one component of an AMS pair process approximating the other is the same as that of the induced stationary mean, which can be described in terms of the time 0 samples of the two random processes.

5.4 Communications Systems Performance

The primary application of the idea of distortion is to the quantification of quality or fidelity in a communications system. Suppose that $[\mu, f, \nu, g]$ is a communications system with overall input/output process is $\{X_n, \hat{X}_n\}$ and alphabet $A \times \hat{A}$. Let p denote the corresponding distribution of the pair process comprised of the input and output. As in Section 5.3, a natural measure of the (lack of) quality of the output or reproduction signal $\hat{X}$ as an approximation to the original input signal X is given by the average limiting distortion $\Delta(p)$, which in the case of a communications system we call the *performance* of the system. In this case there is much going on between the original input and the final output, but it is still the pair process $\{X_n, \hat{X}_n\}$ that determines the performance of the system. Note that in the communications example, the input/output process can be N-stationary if the source source is N-stationary, the first sequence coder (N, K)-stationary, the channel K-stationary (e.g., stationary), and the second sequence coder (K, N)-stationary. It is the overall properties that matter when looking at the performance.

If the source and codes are such that the input/output process is AMS, then the results of Section 5.3 show that the performance satisfies

$$\Delta(p) = \frac{1}{N} E_{\overline{p}_N} \rho_N(X^N, \hat{X}^N) = \frac{1}{N} E_p \rho_\infty^{(N)} = E_{\overline{p}} \rho_1(X_0, \hat{X}_0). \tag{5.13}$$

5.5 Optimal Performance

Given a notion of the performance of a communication system, it makes sense to define the optimal performance achievable when communicating a source $\{X_n\}$ with distribution μ over a channel ν. Suppose that $\mathcal{E}$ is some class of sequence coders $f : A^{\mathbb{T}} \to B^{\mathbb{T}}$. For example, $\mathcal{E}$ might consist of all sequence coders generated by block codes with some constraint or by finite-length sliding-block codes. Similarly let $\mathcal{D}$ denote a class of sequence coders $g : B'^{\mathbb{T}} \to \hat{A}^{\mathbb{T}}$. Define the *operational distortion-rate function (DRF)* for the source μ, channel ν, and code classes $\mathcal{E}$ and $\mathcal{D}$ by

$$\Delta(\mu, \nu, \mathcal{E}, \mathcal{D}) = \inf_{f \in \mathcal{E}, g \in \mathcal{D}} \Delta(\mu, f, \nu, g). \tag{5.14}$$

When the code classes $\mathcal{E}, \mathcal{D}$ are clear from context, the notation is simplified to $\Delta(\mu, \nu)$. When the channel is assumed to be noiseless with alphabet B with $\|B\|$ letters, then all that will matter when considering block and sliding-block codes is the channel rate $R = \log \|B\|$ and the notation is simplified to

$$\delta(R, \mu) = \Delta(\mu, \nu) = \Delta(\mu, \nu, \mathcal{E}, \mathcal{D}). \tag{5.15}$$

When the class of codes being considered is that of sliding-block codes, the operational DRF $\delta(R, \mu)$ will be subscripted as $\delta_{\text{SBC}}(R, \mu)$.

The goal of the coding theorems of information theory is to relate the operational DRF of a source, channel, and code class to (hopefully) computable functions of the source and channel. We will do this in stages in later chapters: first we will focus on the source coding by assuming a noiseless channel, then we will focus on reliable communication over a noisy channel, and lastly we will combine the two.

5.6 Code Approximation

Suppose that $\{X_n\}$ is an information source with process distribution μ and suppose that f and g are two sliding-block codes which share a common reproduction alphabet B. For the moment assume for simplicity that the source is stationary. Let $\mathcal{P}$ and $\mathcal{Q}$ be the corresponding partitions of sequence space, e.g., $\mathcal{P} = \{P_i; i \in \mathbb{I}\}$ where $P_i = f^{-1}(b_i)$. Average distortion can be used to measure how good an approximation one code is for another. This is of interest, for example, if one code is

nearly optimal in some sense, but too complicated to be practical. If another, simpler, code has approximately the same behavior, then it may be a better choice for implementation. Given a distortion measure ρ, the distortion between two codes f, g applied to a common source with distribution μ can be defined as

$$\Delta(f, g) = E_\mu \rho(f, g). \tag{5.16}$$

In the case where both f and g have discrete output alphabets, then a natural distortion is the Hamming distortion and this becomes

$$\Delta_H(f, g) = E_\mu d_H(f, g) = \Pr(f \neq g) = P_e, \tag{5.17}$$

where P_e is a common notation for error probability, the probability that the two discrete random variables f and g differ. This distance between codes can be related to the *partition distance* of ergodic theory between the two partitions $\mathcal{P}$ and $\mathcal{Q}$ which is defined by

$$| \mathcal{P} - \mathcal{Q} | = \sum_{i \in \mathbb{I}} \mu(P_i \Delta Q_i). \tag{5.18}$$

We have that

$$\begin{aligned} \Delta_H(f, g) &= 1 - \Pr(f = g) \\ &= 1 - \sum_i \mu(P_i \cap Q_i) \\ &= \frac{1}{2} \sum_i (\mu(P_i) + \mu(Q_i) - \mu(P_i \cap Q_i)) \\ &= \frac{1}{2} \sum_i \mu(P_i \Delta Q_i) = \frac{1}{2} | \mathcal{P} - \mathcal{Q} | . \end{aligned} \tag{5.19}$$

So far we have considered only a single output of the code, which suffices if the source is stationary. In general, however, we may wish to consider AMS sources and an additive fidelity criterion based on the Hamming distance, in which case the mean distortion is given by

$$\frac{1}{n} \sum_{i=0}^{n-1} \Pr(f_n \neq g_n),$$

where as usual $f_n = fT^n$, and its limit as $n \to \infty$ are of interest. Since stationary codings of an AMS source are jointly AMS, this average converges and we can define a code distance

$$\Delta_H(f, g) = \overline{P}_e = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \Pr(f_n \neq g_n). \tag{5.20}$$

If the source is stationary, then from the stationarity of sliding-block codes this simplifies to $P_e = \Pr(f \neq g)$.

The next lemma and corollary provide tools for approximating complicated codes by simpler ones.

Lemma 5.1. *Given a probability space $(\Omega, \mathcal{B}, P)$ suppose that $\mathcal{F}$ is a generating field: $\mathcal{B} = \sigma(\mathcal{F})$. Suppose that $\mathcal{B}$-measurable $\mathcal{Q}$ is a partition of Ω and $\epsilon > 0$. Then there is a partition $\mathcal{Q}'$ with atoms in $\mathcal{F}$ such that $|\mathcal{Q} - \mathcal{Q}'| \leq \epsilon$.*

Proof: Let $\|A\| = K$. From Theorem 1.1 given $\gamma > 0$ we can find sets $R_i \in \mathcal{F}$ such that $P(Q_i \Delta R_i) \leq \gamma$ for $i = 1, 2, \cdots, K-1$. The remainder of the proof consists of set theoretic manipulations showing that we can construct the desired partition from the R_i by removing overlapping pieces. The algebra is given for completeness, but it can be skipped. Form a partition from the sets as

$$Q'_i = R_i - \bigcup_{j=1}^{i-1} R_j, i = 1, 2, \cdots, K-1$$
$$Q'_K = (\bigcup_{i=1}^{K-1} Q'_i)^c.$$

For $i < K$

$$\begin{aligned} P(Q_i \Delta Q'_i) &= P(Q_i \bigcup Q'_i) - P(Q_i \bigcap Q'_i) \\ &\leq P(Q_i \bigcup R_i) - P(Q_i \bigcap (R_i - \bigcup_{j<i} R_j)). \end{aligned} \tag{5.21}$$

The rightmost term can be written as

$$\begin{aligned} P(Q_i \bigcap (R_i - \bigcup_{j<i} R_j)) &= P((Q_i \bigcap R_i) - (\bigcup_{j<i} Q_i \bigcap R_i \bigcap R_j)) \\ &= P(Q_i \bigcap R_i) - P(\bigcup_{j<i} Q_i \bigcap R_i \bigcap R_j), \end{aligned} \tag{5.22}$$

where we have used the fact that a set difference is unchanged if the portion being removed is intersected with the set it is being removed from and we have used the fact that $P(F - G) = P(F) - P(G)$ if $G \subset F$. Combining (5.21) and (5.22) we have that

$$\begin{aligned} P(Q_i \Delta Q'_i) &\le P(Q_i \bigcup R_i) - P(Q_i \bigcap R_i) + P(\bigcup_{j<i} Q_i \bigcap R_i \bigcap R_j) \\ &= P(Q_i \Delta R_i) + P(\bigcup_{j<i} Q_i \bigcap R_i \bigcap R_j) \\ &\le \gamma + \sum_{j<i} P(Q_i \bigcap R_j). \end{aligned}$$

For $j \neq i$, however, we have that

$$\begin{aligned} P(Q_i \bigcap R_j) &= P(Q_i \bigcap R_j \bigcap Q_j^c) \le P(R_j \bigcap Q_j^c) \\ &\le P(R_j \Delta Q_j) \le \gamma, \end{aligned}$$

which with the previous equation implies that

$$P(Q_i \Delta Q'_i) \le K\gamma; i = 1, 2, \cdots, K-1.$$

For the remaining atom:

$$P(Q_K \Delta Q'_K) = P(Q_K \bigcap Q'^c_K \bigcup Q_K^c \bigcap Q'_K). \tag{5.23}$$

We have

$$Q_K \bigcap Q'^c_K = Q_K \bigcap (\bigcup_{j<K} Q'_j) = Q_K \bigcap (\bigcup_{j<K} Q'_j \bigcap Q_j^c),$$

where the last equality follows since points in Q'_j that are also in Q_j cannot contribute to the intersection with Q_K since the Q_j are disjoint. Since $Q'_j \bigcap Q_j^c \subset Q'_j \Delta Q_j$ we have

$$Q_K \bigcap Q'^c_K \subset Q_K \bigcap (\bigcup_{j<K} Q'_j \Delta Q_j) \subset \bigcup_{j<K} Q'_j \Delta Q_j.$$

A similar argument shows that

$$Q_K^c \bigcap Q'_K \subset \bigcup_{j<k} Q'_j \Delta Q_j$$

and hence with (5.23)

$$P(Q_K \Delta Q'_K) \le P(\bigcup_{j<K} Q_j \Delta Q'_j) \le \sum_{j<K} P(Q_j \Delta Q'_j) \le K^2 \gamma.$$

To summarize, we have shown that

$$P(Q_i \Delta Q'_i) \le K^2 \gamma;\ i = 1, 2, \cdots, K.$$

Choosing γ so small that $K^2\gamma \le \epsilon/K$, the lemma is proved. □

Corollary 5.1. *Let $(\Omega, \mathcal{B}, P)$ be a probability space and $\mathcal{F}$ a generating field. Let $f: \Omega \to A$ be a finite alphabet measurement. Given $\epsilon > 0$ there is a measurement $g: \Omega \to A$ that is measurable with respect to $\mathcal{F}$ (that is, $g^{-1}(a) \in \mathcal{F}$ for all $a \in A$) for which $\Delta_H(f, g) = \Pr(f \neq g) \leq \epsilon$.*

Proof: Follows from the previous lemma by setting $\mathcal{Q} = \{f^{-1}(a);\ a \in A\}$, choosing $\mathcal{Q}'$ from the lemma, and then assigning g for atom Q'_i in $\mathcal{Q}'$ the same value that f takes on in atom Q_i in $\mathcal{Q}$. Then

$$\Pr(f \neq g) = \frac{1}{2} \sum_i P(Q_i \Delta Q'_i) = \frac{1}{2} \mid \mathcal{Q} - \mathcal{Q}' \mid \leq \epsilon.$$

□

A stationary code f is a *scalar quantizer* if there is a map $q: A_X \to A_f$ such that $f(x) = q(x_0)$. Intuitively, f depends on the input sequence only through the current symbol. Mathematically, f is measurable with respect to $\sigma(X_0)$. Such codes are effectively the simplest possible and have no memory or dependence on the future.

Lemma 5.2. *Let $\{X_n\}$ be an AMS process with standard alphabet A_X and distribution μ. Let f be a stationary coding of the process with finite alphabet A_f. Fix $\epsilon > 0$. If the process is two-sided, then there is a scalar quantizer $q: A_X \to A_q$, an integer N, and a mapping $g: A_q^N \to A_f$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \Pr(f_i \neq g(q(X_{i-N}), q(X_{i-N+1}), \cdots, q(X_{i+N}))) \leq \epsilon.$$

If the process is one-sided, then there is a scalar quantizer $q: A_X \to A_q$, an integer N, and a mapping $g: A_q^N \to A_f$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \Pr(f_i \neq g(q(X_i), q(X_{i+1}), \cdots, q(X_{i+N-1}))) \leq \epsilon.$$

Comment: The lemma states that any stationary coding of an AMS process can be approximated by a code that depends only on a finite number of quantized inputs, that is, by a coding of a finite window of a scalar quantized version of the original process. In the special case of a finite alphabet input process, the lemma states that an arbitrary stationary coding can be well approximated by a coding depending only on a finite number of the input symbols.

Proof: Suppose that $\overline{m}$ is the stationary mean and hence for any measurements f and g

$$\overline{m}(f_0 \neq g_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{n=0}^{n-1} \Pr(f_i \neq g_i).$$

Let q_n be an asymptotically accurate scalar quantizer in the sense that $\sigma(q_n(X_0))$ asymptotically generates $\mathcal{B}(A_X)$. (Since A_X is standard this exists. If A_X is finite, then take $q(a) = a$.) Then the finite fields

$$\mathcal{F}_n = \sigma(q_n(X_i); i = 0, 1, 2, \cdots, n-1) \tag{5.24}$$

asymptotically generates $\mathcal{B}(A_X)^\infty$ for one-sided processes and

$$\mathcal{F}_n = \sigma(q_n(X_i);\ i = -n, \cdots, n) \tag{5.25}$$

does the same for two-sided processes. Hence from Corollary 5.1 given ϵ we can find a sufficiently large n and a mapping g that is measurable with respect to $\mathcal{F}_n$ such that $\overline{m}(f \neq g) \leq \epsilon$. Since g is measurable with respect to $\mathcal{F}_n$, it must depend on only the finite number of quantized samples that generate $\mathcal{F}_n$. (See, e.g., Lemma 5.2.1 of [55] or Lemma 6.1 of [58].) This proves the lemma. □

5.7 Approximating Random Vectors and Processes

The material in this section draws heavily on Section 9.2 of [58], where proofs and further details may be found.

In the previous sections it was pointed out that if one has a distortion measure ρ on two random objects X and Y and a joint distribution on the two random objects (and hence also marginal distributions for each), then a natural notion of the difference between the random objects or the poorness of their mutual approximation is the expected distortion $E\rho(X, Y)$. We now consider a different question: What if one does not have a joint probabilistic description of X and Y, but instead knows only their marginal distributions. What then is a natural notion of the distortion or poorness of approximation of the two random objects? In other words, we previously measured the distortion between two random objects whose stochastic connection was determined, possibly by a channel, a code, or a communication system. We now wish to find a similar quantity for the case when the two random objects are only described as individuals. One possible definition is to find the smallest possible distortion in the old sense consistent with the given information, that is, to minimize $E\rho(X, Y)$ over all couplings of the marginal distributions μ_X and μ_Y; that is, the minimization over all joint distributions consistent with the given marginal distributions. Note that this will necessarily give a lower bound to the distortion achievable when any specific joint distribution is specified.

To be precise, suppose that we have random objects X and Y with distributions μ_X and μ_Y and alphabets A and B, respectively. Let ρ be a distortion measure on $A \times B$. Define the *$\overline{\rho}$-distortion* (pronounced ρ-bar

and also written as "rho-bar") between the random objects X and Y by

$$\overline{\rho}(\mu_X, \mu_Y) = \inf_{\pi \in \mathcal{P}} E_p \rho(X, Y), \tag{5.26}$$

where $\mathcal{P} = \mathcal{P}(\mu_X, \mu_Y)$ is the collection of all measures on $(A \times B, \mathcal{B}_A \times \mathcal{B}_B)$ with μ_X and μ_Y as marginals; that is, if $\pi \in \mathcal{P}$, then

$$\pi(A \times F) = \mu_Y(F);\ F \in \mathcal{B}_B; \pi(G \times B) = \mu_X(G);\ G \in \mathcal{B}_A. \tag{5.27}$$

As observed in Section 2.21, $\mathcal{P}$ is not empty since, for example, it contains the product measure $\mu_X \times \mu_Y$, so the optimization is well defined.

The above notation emphasizes that rho-bar measures the distortion between two probability distributions μ_X and μ_Y. As with the various notations for entropy and entropy rate, it is often convenient to consider the distortion as a measure of the difference of the random variables rather than their distributions and write $\overline{\rho}(X, Y)$ for $\overline{\rho}(\mu_X, \mu_Y)$. The random variable notation has the advantage of economy, but both forms will be used.

This definition is more suited to random variables and vectors than to random processes, but before extending the definition to processes some historical remarks are in order.

The above formulation dates back to Kantorovich's minimization of average cost over couplings with prescribed marginals [85]. Kantorovich considered compact metric spaces with metric cost functions. The modern literature considers cost functions that are essentially as general as the distortion measures considered in information theoretic coding theorems. Results exist for general distortion measures/cost functions, but stronger and more useful results exist when the structure of metric spaces is added by concentrating on distortion measures which are positive powers of metrics.

The problem for Kantorovich arose in the study of resource allocation in economics, work for which he later shared the 1974 Nobel prize for economics. The problem is an early example of linear programming, and Kantorovich is generally recognized as an inventor of linear programming. Kantorovich later realized [86] that his problem was a generalization of a 1781 problem of Monge [125], which can be described in in the current context as follows. Fix as before the marginal distributions μ_X, μ_Y. Given a measurable mapping $f : A \to B$, define as usual the induced measure $\mu_X f^{-1}$ by

$$\mu_X f^{-1}(G) = \mu_X(f^{-1}(G)); G \in \mathcal{B}_B. \tag{5.28}$$

Define

$$\tilde{\rho}(\mu_X, \mu_Y) = \inf_{f:\mu_X f^{-1} = \mu_Y} E_{\mu_X} \rho(X, f(X)). \tag{5.29}$$

In this formulation the coupling of two random objects by a joint measure is replaced by a deterministic mapping of the first into the second. This effectively constrains the optimization to the special case of deterministic mappings and the resulting optimization is no longer one with linear constraints, which results in increased difficulty of analysis and evaluation.

The problem of (5.26) is widely known as the Monge/Kantorovich *optimal transportation* or *transport* problem, and when the cost function is a metric the resulting minimum average distortion has been called the Monge/Kantorovich distance. There is also a more general meaning of the name "Monge/Kantorovich distance," as will be discussed. While it is the Kantorovich formulation that is most relevant here, the Monge formulation provides an interesting analogy when coding schemes are introduced since the mapping f can be interpreted as a coding of X into Y. A significant portion of the optimal transportation literature deals with conditions under which the two optimizations yield the same values and the optimal coupling is a deterministic mapping.

When the idea was rediscovered in 1974 [137, 66] as part of an extension of Ornstein's $\overline{d}$-distance for random processes [138, 139, 140] (more about this later), it was called the $\overline{\rho}$ or "rho-bar distance." It was recognized, however, that the finite dimensional case (random vectors) was Vasershtein's distance [183], which had been popularized by Dobrushin [34], and, in [137], as the transportation distance. In the optimal transportation literature, the German spelling of Vasershtein, Wasserstein, also caught on as a suitable name for the Monge/Kantorovich distance and remains so.

The Monge/Kantorovich optimal cost provides a distortion or distance between measures, random variables, and random vectors. It has found a wide variety of applications and there are several thorough modern books detailing the theory and applications. The interested reader is referred to [153, 154, 185, 186, 187] and the enormous number of references therein. The richness of the field is amply demonstrated by the more than 700 references cited in [187] alone! Many examples of evaluation, dual formulations, and methods for constructing the measures and functions involved have appeared in the literature. Names associated with the distance include Gini [51], Fréchet [45], Vasershtein/Wasserstein [183], Vallender [181], Dobrushin [34], Mallows [116], and Bickel and Freedman [15]. It was rediscovered in the computer science literature as the "earth mover's distance" with an intuitive description close to that of the original transportation distance [159, 108].

Because of the plethora of contributing authors from different fields, the distortion presents a challenge of choosing both notation and a name. Because this book shares the emphasis on random processes with Ornstein and ergodic theory, the notation and nomenclature will generally follow the ergodic theory literature and the derivative information

theory literature. Initially, however, we adopt the traditional names for the traditional setting, and much of the notation in the various fields is similar.

5.8 The Monge/Kantorovich/Vasershtein Distance

Assume that both μ_X and μ_Y are distributions on random objects X and Y defined on a common Polish space (A, d). Define the distortion measure $\rho = d^p$ for $p \geq 0$. The $p = 0$ case is shorthand for the Hamming distance, that is,

$$d^0(x, y) = d_H(x, y) = 1 - \delta_{x,y}. \tag{5.30}$$

Define the collection of measures $\mathcal{P}_p(A, \mathcal{B}(A))$ as the collection of all probability measures μ on $(A, \mathcal{B}(A))$ for which there exists a point $a^* \in A$ such that

$$\int d^p(x, a^*) d\mu(x) < \infty. \tag{5.31}$$

The point a^* is called a *reference letter* in information theory [47].

Define

$$\overline{d}_p(\mu_X, \mu_Y) = \overline{\rho}(\mu_X, \mu_Y)^{\min(1,1/p)} = \begin{cases} \overline{\rho}(\mu_X, \mu_Y) & 0 \leq p \leq 1 \\ \overline{\rho}^{1/p}(\mu_X, \mu_Y) & 1 \leq p \end{cases}. \tag{5.32}$$

The following lemma is Lemma 9.4 from [58]. It shows that $\overline{d}_p$ is indeed a metric on $\mathcal{P}_p(A, \mathcal{B}(A))$.

Lemma 5.3. *Given a Polish space (A, d) with a Borel σ-field $\mathcal{B}(A)$, let $\mathcal{P}_p(A, \mathcal{B}(A))$ denote the collection of all Borel probability measures on the Borel measurable space $(A, \mathcal{B}(A))$ with a reference letter. Then for any $p \geq 0$, $\overline{d}_p$ is a metric on $\mathcal{P}_p(A, \mathcal{B}(A))$.*

5.9 Variation and Distribution Distance

Two other notions of distance between random vectors or their distributions μ_X and μ_Y are useful for comparisons: the *variation distance* (or *variation* or *variational distance*) and the *distribution distance*. Unfortunately the notation in the literature is not consistent, with a factor of 2 often being present or not.

Pinsker [150] defined the *variation* of μ_X with respect to μ_Y by

$$\mathrm{var}(\mu_X, \mu_Y) = \sup_{\mathcal{P}} \sum_i \mid \mu_X(P_i) - \mu_Y(P_i) \mid \tag{5.33}$$

where the supremum is over all partitions $\mathcal{P} = \{P_i\}$. This is often called the *variation distance* between the two probability distributions, but unfortunately the usage is not uniform. In the case of discrete distributions, the variation distance specializes to the *distribution distance* defined as the ℓ_q norm between the probability mass functions

$$\mid \mu_X - \mu_Y \mid = \sum_{x \in A} \mid \mu_X(x) - \mu_Y(x) \mid .$$

This is easy to see since the partition of the space into points yields

$$\begin{aligned} \mathrm{var}(\mu_X, \mu_Y) &= \sup_{\mathcal{P}} \sum_i \mid \mu_X(P_i) - \mu_Y(P_i) \mid \\ &\geq \sum_{x \in A} \mid \mu_X(x) - \mu_Y(x) \mid = \mid \mu_X - \mu_Y \mid \end{aligned}$$

and, conversely, given any atom P of any partition,

$$\begin{aligned} \mid \mu_X(P) - \mu_Y(P) \mid &= \mid \sum_{a \in P} \mu_X(a) - \sum_{a \in P} \mu_Y(a) \mid \\ &= \mid \sum_{a \in P} (\mu_X(a) - \mu_Y(a)) \mid \\ &\leq \sum_{a \in P} \mid \mu_X(a) - \mu_Y(a)) \mid . \end{aligned}$$

Unfortunately a slightly different notion of distance is often given the same name of variation or variational distance. It also has a separate name of *total variation* or *total variation distance*, which we will adopt:

$$\mathrm{tvar}(\mu_X, \mu_Y) = \sup_G \mid \mu_X(G) - \mu_Y(G) \mid, \tag{5.34}$$

where the supremum is over all events G. The total variation distance quantifies the notion of the maximum possible difference between what two measures can assign to a single event. The variation and total variation distances are related by a factor of 2, as is made precise in the following lemma.

Lemma 5.4. *Given two probability measures M, P on a common measurable space $(\Omega, \mathcal{B})$,*

$$\mathrm{var}(P, M) = 2\,\mathrm{tvar}(P, M). \tag{5.35}$$

Proof. First observe that for any set F we have for the partition $\mathcal{Q} = \{F, F^c\}$ that

$$\mathrm{var}(P,M) \geq \sum_{Q\in\mathcal{Q}} |P(Q) - M(Q)| = 2|P(F) - M(F)|$$

and hence

$$\mathrm{var}(P,M) \geq 2 \sup_{F\in\mathcal{B}} |P(F) - M(F)| = \mathrm{tvar}(P,M)$$

Conversely, suppose that $\mathcal{Q}$ is a partition which approximately yields the variational distance, e.g.,

$$\sum_{Q\in\mathcal{Q}} |P(Q) - M(Q)| \geq \mathrm{var}(P,M) - \epsilon$$

for $\epsilon > 0$. Define a set F as the union of all of the Q in $\mathcal{Q}$ for which $P(Q) \geq M(Q)$ and we have that

$$\sum_{Q\in\mathcal{Q}} |P(Q) - M(Q)| = P(F) - M(F) + M(F^c) - P(F^c) = 2(P(F) - M(F))$$

and hence

$$\mathrm{var}(P,M) - \epsilon \leq \sup_{F\in\mathcal{B}} 2|P(F) - M(F)|.$$

Since ϵ is arbitrary, this proves the first statement of the lemma. □

The reader should be wary of the factor of 2 in the two definitions of "variation distance" as it results in different statements of Pinsker's inequality in the literature.

5.10 Coupling Discrete Spaces with the Hamming Distance

Dobrushin [34] developed several interesting properties for the special case of the transportation or Monge/Kantorovich distance, which he called the Vasershtein distance [183], with a Hamming distortion. The properties prove useful when developing examples of the process distance, so they are collected here.

For this section we consider probability measures μ_X and μ_Y on a common discrete space A. We will use μ_X to denote both the measure and the PMF, e.g., $\mu_X(G)$ is the probability of a subset $G \subset S$ and $\mu(x) = \mu(\{x\})$ is the probability of the one point set $\{x\}$. Specifically, X and Y are random objects with alphabets $A_X = A_Y = A$. Let (X,Y) denote the identity mapping on the product space $A \times A$ and let X and Y denote the projections onto the component spaces given by $X(x,y) = x$, $Y(x,y) = y$. Denote as before the collection of all couplings π satisfying (5.27) by $\mathcal{P}(\mu_X, \mu_Y)$.

The Hamming transportation distance is given by

$$\overline{d}_H(\mu_X, \mu_Y) = \inf_{\pi \in \mathcal{P}(\mu_X, \mu_Y)} d_H(X, Y) = \inf_{\pi \in \mathcal{P}(\mu_X, \mu_Y)} \pi(X \neq Y).$$

The following is Lemma 9.5 of [58] stated in the vocabulary adopted here.

Lemma 5.5. *For discrete distributions μ_X, μ_Y*

$$\overline{d}_H(\mu_X, \mu_Y) = \text{tvar}(\mu_X, \mu_Y) = \frac{1}{2}\text{var}(\mu_X, \mu_Y) = \frac{1}{2} \mid \mu_X - \mu_Y \mid .$$

5.11 Process Distance and Approximation

In information theory, ergodic theory, and signal processing, it is important to quantify the distortion between *processes* rather than only between probability distributions, random variables, or random vectors. This means that instead of having a single space of interest, one is interested in a limit of finite-dimensional spaces or a single infinite-dimensional space. In this case one needs a family of distortion measures for the finite-dimensional spaces which will yield a notion of distortion and optimal coupling for the entire process. We have seen in Section 5.3 one way to do this when measuring distortion between two components of a pair random process. In this section the topic is explored further with the goal of finding a useful measure of distortion or distance between two random processes.

We shall focus on additive fidelity criteria which define the per-letter distortion ρ_1 in terms of an underlying metric d. The most commonly occurring examples are to set $\rho_1(x, y) = d(x, y)$, in which case ρ_n is also a metric for all n, or $\rho_1(x, y) = d^p(x, y)$ for some $p > 0$. If $1 > p > 0$, then again ρ_n is a metric for all n. If $p \geq 1$, then ρ_n is not a metric, but $\rho_n^{1/p}$ is a metric. We do not wish to include the $1/p$ in the definition of the fidelity criterion, however, because what we gain by having a metric distortion we more than lose when we take the expectation. For example, a popular distortion measure is the expected squared error, not the expectation of the square root of the squared error. Thanks to Lemma 5.3 we can still get the benefits of a distance even with the nonmetric distortion measure, however, by taking the appropriate root outside of the expectation.

A small fraction of the information theory and communications literature consider the case of $\rho_1 = f(d)$ for a nonnegative convex function f, which is more general than $\rho_1 = d^p$ if $p \geq 1$. We will concentrate on the simpler case of a positive power of a metric.

For the time being we will make the following assumptions:

- $\{\rho_n; n = 1, 2, \ldots\}$ is an additive fidelity criterion defined by

$$\rho_n(x^n, y^n) = \sum_{i=0}^{n-1} d^p(x_i, y_i), \tag{5.36}$$

  where d is a metric and $p > 0$. We will refer to this as the d^p-distortion. We will also allow the case $p = 0$ which is defined by

$$\rho_1(x, y) = d^0(x, y) = d_H(x, y),$$

  the Hamming distance $d_H(x, y) = 1 - \delta_{x,y}$.
- The random process distributions μ considered will possess a *reference letter* in the sense of Gallager [47]: given a stationary μ it is assumed that there exists an $a \in A$ for which

$$\int d^p(x, a^*) d\mu^1(x) < \infty. \tag{5.37}$$

  If the process is not stationary, we assume that there exists an $a \in A$ and $\rho^* < \infty$ such that for all marginal distributions μ_n^1

$$\int d^p(x, a^*) d\mu_n^1(x) < \rho^* \tag{5.38}$$

  so that one reference letter works for all times n. This is trivially satisfied if the distortion is bounded.

Define $\overline{\mathcal{P}}_p(A, d)$ to be the space of all stationary process distributions μ on $(A, \mathcal{B}(A))^{\mathbb{T}}$ satisfying (5.37), where both $\mathbb{T} = \mathbb{Z}$ and $\mathbb{Z}_+$ will be considered. Note that previously $\mathcal{P}_p(A, d)$ referred to a space of distributions of random objects taking values in a metric space A. Now it refers to a space of process distributions with all individual component random variables taking values in a common metric space A, that is, the process distributions are on $(A, \mathcal{B}(A))^{\mathbb{T}}$.

In the process case we will consider Monge/Kantorovich distances on the spaces of random vectors with alphabets A^n for all $n = 1, 2, \ldots$, but we shall define a process distortion and metric as the supremum over all possible vector dimensions.

Given two process distributions μ_X and μ_Y describing random processes $\{X_n\}$ and $\{Y_n\}$, let the induced n-dimensional distributions be μ_{X^n}, μ_{Y^n}. For each positive integer n we can define the n-dimensional or n-th order optimal coupling distortion as the distortion between the induced n-dimensional distributions:

$$\overline{\rho}_n(\mu_{X^n}, \mu_{Y^n}) = \inf_{\pi \in \mathcal{P}_p(\mu_{X^n}, \mu_{Y^n})} E_\pi \rho_n(X^n, Y^n) \tag{5.39}$$

The process optimal coupling distortion (or $\overline{\rho}$ distortion) between μ_X and μ_Y is defined by

$$\overline{\rho}(\mu_X, \mu_Y) = \sup_n \frac{1}{n} \overline{\rho}_n(\mu_{X^n}, \mu_{Y^n}). \tag{5.40}$$

The extension of the optimal coupling distortion or transportation cost to processes was developed in the early 1970s by D.S. Ornstein for the case of $\rho_1 = d_H$ and the resulting process metric, called the $\overline{d}$ distance or d-bar distance or Ornstein distance, played a fundamental role in the development of the Ornstein isomorphism theorem of ergodic theory (see [138, 139, 140] and the references therein). The idea was extended to processes with Polish alphabets and metric distortion measures and the square of metric distortion measures in 1975 [66] and applied to problems of quantizer mismatch [60], Shannon information theory [63, 132, 64, 52, 67, 53], and robust statistics [142]. While there is a large literature on the finite-dimensional optimal coupling distance, the literature for the process optimal coupling distance seems limited to the information theory and ergodic theory literature. Here the focus is on the process case, but the relevant finite-dimensional results are also treated as needed.

The key aspect of the process distortion is that if it is small, then necessarily the distortion between all sample vectors produced by the two processes is also small.

The $\overline{d}_p$-distance

The process distortion is a metric for the important special case of an additive fidelity criterion with a metric per letter distortion. This subsection shows how a process metric can be obtained in the most important special case of an additive fidelity criterion with a per letter distortion given by a positive power of a metric. The result generalizes the special cases of process metrics in [139, 66, 179] and extends the well known finite-dimensional version of optimal transportation theory (e.g., Theorem 7.1 in [186]).

Theorem 5.1. *Given a Polish space (A, d) and $p \geq 0$, define the additive fidelity criterion $\rho_n : A^n \times A^n \to \mathbb{R}_+; n = 1, 2, \ldots$ by*

$$\rho_n(x^n, y^n) = \sum_{n=0}^{n-1} d^p(x_i, y_i),$$

where d^0 is shorthand for the Hamming distance d_H. Define

$$\overline{d}_p(\mu_X, \mu_Y) = \sup_n n^{-1}\overline{\rho}_n^{\min(1,1/p)}(\mu_{X^n}, \mu_{Y^n}) = \overline{\rho}^{\min(1,1/p)}(\mu_X, \mu_Y). \quad (5.41)$$

Then $\overline{d}_p$ is a metric on $\overline{\mathcal{P}}_p(A, d)$, the space of all stationary random processes with alphabet A.

The theorem together with Lemma 5.3 says that if $\rho_1 = d^p$, then $\overline{d}_p = \overline{\rho}^{1/p}$ is a metric for both the vector and process case if $p \geq 1$, and $\overline{d}_p = \overline{\rho}$ is a metric if $0 \leq p \leq 1$. The two cases agree for $p = 1$ and the $p = 0$ case is simply shorthand for the $p = 1$ case with the Hamming metric. In the case of $p = 0$, the process distance $\overline{d}_0$ is Ornstein's $\overline{d}$-distance and the notation is usually abbreviated to simply $\overline{d}$ to match usage in the ergodic theory literature. It merits pointing out that $\overline{d}_0(\mu_{X^n}, \mu_{Y^n})$ is *not* the transportation distance with a Hamming distance on A^n, it is the transportation distance with respect to the distance $d(x^n, y^n) = \sum_{i=0}^{n-1} d_H(x_i, y_i)$, the sum of the Hamming distances between symbols. This is also n times the average Hamming distance. The two metrics on A^n are related through the simple bounds

$$\sum_{i=0}^{n-1} d_H(x_i, y_i) \geq d_H(x^n, y^n) \geq \frac{1}{n}\sum_{i=0}^{n-1} d_H(x_i, y_i). \quad (5.42)$$

The next theorem collects several more properties of the $\overline{d}_p$ distance between stationary processes, including the facts that the supremum defining the process distance is a limit, that the distance between IID processes reduces to the Monge/Kantorovich distance between the first order marginals, and a characterization of the process distance as an optimization over processes.

Theorem 5.2. *Suppose that μ_X and μ_Y are stationary process distributions with a common standard alphabet A and that $\rho_1 = d^p$ is a positive power of a metric on A and that ρ_n is defined on A^n in an additive fashion as before. Then*

(a) *$\lim_{n\to\infty} n^{-1}\overline{\rho}_n(\mu_{X^n}, \mu_{Y^n})$ exists and equals $\sup_n n^{-1}\overline{\rho}_n(\mu_{X^n}, \mu_{Y^n})$. Thus $\overline{d}_p(\mu_X, \mu_Y) = \lim_{n\to\infty} n^{-1}\overline{\rho}_n^{\min(1,1/p)}(\mu_{X^n}, \mu_{Y^n})$.*

(b) *If μ_X and μ_Y are both IID, then $\overline{\rho}(\mu_X, \mu_Y) = \overline{\rho}_1(\mu_{X_0}, \mu_{Y_0})$ and hence $\overline{d}_p(\mu_X, \mu_Y) = \overline{\rho}_1^{\min(1,1/p)}(\mu_{X_0}, \mu_{Y_0})$*

(c) *Let $\mathcal{P}_s = \mathcal{P}_s(\mu_X, \mu_Y)$ denote the collection of all stationary distributions π_{XY} having μ_X and μ_Y as marginals, that is, distributions on $\{X_n, Y_n\}$ with coordinate processes $\{X_n\}$ and $\{Y_n\}$ having the given distributions. Define the process distortion measure $\overline{\rho}'$*

$$\overline{\rho}'(\mu_X, \mu_Y) = \inf_{\pi_{XY}\in\mathcal{P}_s} E_{\pi_{XY}}\rho(X_0, Y_0).$$

Then

$$\overline{\rho}(\mu_X, \mu_Y) = \overline{\rho}'(\mu_X, \mu_Y);$$

that is, the limit of the finite dimensional minimizations is given by a minimization over stationary processes.

(d) *Suppose that μ_X and μ_Y are both stationary and ergodic. Define $\mathcal{P}_e = \mathcal{P}_e(\mu_X, \mu_Y)$ as the subset of $\mathcal{P}_s$ containing only ergodic processes, then*

$$\overline{\rho}(\mu_X, \mu_Y) = \inf_{\pi_{XY} \in \mathcal{P}_e} E_{\pi_{XY}} \rho(X_0, Y_0).$$

(e) *Suppose that μ_X and μ_Y are both stationary and ergodic. Let G_X denote a collection of frequency-typical or generic sequences for μ_X in the sense of Section 8.3 of [55] or Section 7.9 of [58]. Frequency-typical sequences are those along which the relative frequencies of a set of generating events all converge and hence by measuring relative frequencies on frequency-typical sequences one can deduce the underlying stationary and ergodic measure that produced the sequence. An AMS process produces frequency-typical sequences with probability 1. Similarly let G_Y denote a set of frequency-typical sequences for μ_Y. Define the process distortion measure*

$$\overline{\rho}''(\mu_X, \mu_Y) = \inf_{x \in G_X, y \in G_Y} \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \rho_1(x_0, y_0).$$

Then

$$\overline{\rho}(\mu_X, \mu_Y) = \overline{\rho}''(\mu_X, \mu_Y).$$

that is, the $\overline{\rho}$ distortion gives the minimum long term time average distance obtainable between frequency-typical sequences from the two processes.

Proofs of these results can be found in Section 9.4 of [58], but the proof of part (c) is not correct. For completeness the proof is presented here.

Proof. **(c)**

(c) Given $\epsilon > 0$ let $\pi \in \mathcal{P}_s(\mu_X, \mu_Y)$ be such that $E_\pi \rho_1(X_0, Y_0) \leq \overline{\rho}'(\mu_X, \mu_Y) + \epsilon$. The induced distribution on $\{X^n, Y^n\}$ is then contained in $\mathcal{P}_n(\mu_{X^n}, \mu_{Y^n})$, and hence using the stationarity of the processes

$$\overline{\rho}_n(\mu_{X^n}, \mu_{Y^n}) \leq E\rho_n(X^n, Y^n) = nE\rho_1(X_0, Y_0) \leq n(\overline{\rho}'(\mu_X, \mu_Y) + \epsilon,$$

and therefore $\overline{\rho}' \geq \overline{\rho}$ since ϵ is arbitrary.

Let $\pi^n \in \mathcal{P}_n,\ n = 1, 2, \ldots$ be a sequence of measures such that

$$E_{\pi^n}[\rho_n(X^n, Y^n)] \leq \overline{\rho}_n + \epsilon_n$$

where $\epsilon_n > 0$ and $\epsilon_n \to 0$ as $n \to \infty$. Let q_n denote the product (block independent) measure $(A^{\mathbb{T}}, \mathcal{B}(A)^{\mathbb{T}})^2$ induced by the π^n as explained

next. Let $\mathcal{G}$ denote a countable generating field for the standard space $(A, \mathcal{B}(A))$. For any N and N-dimensional rectangle or cylinder of the form $F = \times_{i\in\mathbb{T}} F_i$ with all but a finite number N of the F_i being A^2 and the remainder being in $\mathcal{G}^2$ define

$$q_n(F) = \prod_{j\in\mathbb{T}} \pi^n(F_{jn} \times F_{jn+1} \times \cdots \times F_{jn+n-1}).$$

Thus q_n assigns a probability to rectangles in a way that treats successive n-tuples as independent. Next "stationarize" q_n to form a measure on rectangles by averaging over n-shifts to form

$$\pi_n(F) = \frac{1}{n}\sum_{i=0}^{n-1} q_n(T^{-i}F) = \frac{1}{n}\sum_{i=0}^{n-1}\prod_{j\in\mathbb{T}} \pi^n(F_{jn+i}\times F_{jn+i+1}\times\cdots\times F_{jn+i+n-1}).$$

This measure on the rectangles extends to a stationary pair process distribution. For any $m = 1, 2, \ldots, n$ we can relate the mth marginal restrictions of π_n to the corresponding original marginals. For example, consider the Y marginal and let $G = \times_{k=0}^{m-1} G_i \in \mathcal{G}^m$. Then

$$\begin{aligned} q_n(\{x, y : x^m \in A^m, y^m \in G) &= \pi^n(A^n \times (G \times A^{n-m})) \\ &= \mu_{Y^n}(G \times A^{n-m}) = \mu_{Y^m}(G) \end{aligned}$$

and similarly if $G \in \mathcal{B}(A)^m$ then

$$q_n(\{x, y : x^m \in G, y^m \in B^m\}) = \mu_{X^m}(G).$$

Thus

$$\begin{aligned} &\pi_n^m(A^m \times G) \qquad (5.43)\\ &= \pi_n(\{(x, y) : x^m \in A^m, y^m \in G\}) \\ &= \frac{n-m+1}{n}\mu_{Y^m}(G) + \frac{1}{n}\sum_{i=1}^{m-1} \mu_{Y^{m-i}}(\times_{k=i}^{m-i} A)\mu_{Y^i}(\times_{k=0}^{i-1} G_k) \qquad (5.44) \end{aligned}$$

with a similar expression for $G \times A^m$.

Since there are a countable number of finite dimensional rectangles in $\mathcal{B}^{\mathbb{T}}$ with coordinates in $\mathcal{G}$, we can use a diagonalization argument to extract a subsequence π_{n_k} of π_n which converges on all of the rectangles. To do this enumerate all the rectangles, then pick a subsequence converging on the first, then a further subsequence converging on the second, and so on. The result is a limiting measure π on the finite-dimensional rectangles, and this can be extended to a measure also denoted by π on $(A, \mathcal{B}(A))^2$, that is, to a stationary pair process distribution. Eq. (5.44) implies that for each fixed m

$$\lim_{n\to\infty} \pi_n(A^m \times G) = \pi^m(A^m \times G) = \mu_{Y^m}(G)$$
$$\lim_{n\to\infty} \pi_n(G \times A^m) = \pi^m(G \times A^m) = \mu_{X^m}(G)$$

and hence for any cylinder $F \in \mathcal{B}(A)$ that

$$\pi(A^{\mathbb{T}} \times F) = \mu_Y(F)$$
$$\pi(F \times A^{\mathbb{T}}) = \mu_X(F)$$

Thus π induces the desired marginals and hence $\pi \in \mathcal{P}_s$ and

$$\rho'(\mu_X, \mu_Y) \le E_\pi \rho_1(X_0, Y_0) = \lim_{k\to\infty} E_{\pi_{n_k}} \rho_1(X_0, Y_0)$$
$$= \lim_{k\to\infty} n_k^{-1} \sum_{i=0}^{n_k-1} E_{q_{n_k}} \rho_1(X_i, Y_i) = \lim_{k\to\infty} (\overline{\rho}_{n_k} + \epsilon_{n_k}) = \overline{\rho}(\mu_X, \mu_Y).$$

□

Evaluating Process Distortion

Evaluation of the rho-bar distortion or d-bar distance can in general be difficult analytically. Theorem 5.2 provides an important exception, if both of the processes are IID then the process distance is given by the distance between the zero-time samples, the first-order coupling distance. From Lemma 5.5, if the distance is with respect to the Hamming distance, this in turn is given by half the variation distance. In [66] the distance between Gaussian processes was shown to be the L_2 distance between the square roots of their power spectral densities.

5.12 Source Approximation and Codes

In Section 5.6 the approximation of the output of two codes applied to a common source was considered. A natural variation on this idea is to fix a code and look at the approximation of the two outputs resulting from different sources. We again focus on the d-bar distance.

Lemma 5.6. *Let μ_X and μ_Y be distributions of two stationary random processes on a common discrete alphabet, let f be a sliding-block code of length N, and let $\mu_{f(X)}$ and $\mu_{f(Y)}$ denote the corresponding output distributions of coding μ_X or μ_Y with f. Then*

$$\overline{d}(\mu_{f(X)}, \mu_{f(Y)}) \le N\overline{d}(\mu_X, \mu_Y).$$

Proof. If f is a sliding-block code of length N then it depends only on $X_m^N = (X_m, X_{m+1}, \ldots, X_{m+N-1})$ for some fixed m. Choose a coupling of the two processes yielding $\Pr(X_0 \neq Y_0) = \overline{d}(\mu_X, \mu_Y)$. We have not shown that d-bar can be hit with equality like this, but it turns out to be the case (and the following argument works with the addition of a small $\epsilon > 0$). This coupling of input processes implies a coupling of the output processes, so that with a slight abuse of notation we have from the union bound that

$$\begin{aligned}\overline{d}(\mu_{f(X)}, \mu_{f(Y)}) &\leq \Pr(f(X_m^N) \neq f(Y_m^N)) \\ &\leq \Pr(X_m^N \neq Y_m^N) \leq \sum_{i=m}^{m+N-1} \Pr(X_i \neq Y_i) \\ &= N \Pr(X_0 \neq Y_0) = N\overline{d}(\mu_X, \mu_Y).\end{aligned}$$

□

Thus in particular the output of a finite-length sliding-block code is a continuous function of the input in d-bar distance (with respect to the Hamming distance).

5.13 d-bar Continuous Channels

The $\overline{d}$ distance can be used to generalize some of the notions of discrete-alphabet channels by weakening the definitions. The first definition is the most important for channel coding applications. We confine interest to the d-bar or $\overline{d}_0$ distance, the $\overline{\rho}$-distortion for the special case of the Hamming distance:

$$\rho_1(x, y) = d_1(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Suppose that $[A, \nu, B]$ is a discrete alphabet channel and let ν_x^n denote the restriction of the channel to B^n, that is, the output distribution on Y^n given an input sequence x. The channel is said to be *$\overline{d}$-continuous* if for any $\epsilon > 0$ there is an n_0 such that for all $n > n_0$ $\overline{d}_n(\nu_x^n, \nu_{x'}^n) \leq \epsilon$ whenever $x_i = x'_i$ for $i = 0, 1, \cdots, n$. Alternatively, ν is $\overline{d}$-continuous if

$$\limsup_{n \to \infty} \sup_{a^n \in A^n} \sup_{x, x' \in c(a^n)} \overline{d}_n(\nu_x^n, \nu_{x'}^n) = 0,$$

where $c(a^n)$ is the rectangle defined as all x with $x_i = a_i$; $i = 0, 1, \cdots, n-1$. $\overline{d}$-continuity implies the distributions on output n-tuples Y^n given two input sequences are very close provided that the input sequences are identical over the same time period and that n is large.

This generalizes the notions of 0 or finite input memory and anticipation since the distributions need only approximate each other and do not have to be exactly the same.

More generally we could consider $\overline{\rho}$-continuous channels in a similar manner, but we will focus on the simpler discrete $\overline{d}$-continuous channel.

$\overline{d}$-continuous channels possess continuity properties that will be useful for proving block and sliding-block coding theorems. They are "continuous" in the sense that knowing the input with sufficiently high probability for a sufficiently long time also specifies the output with high probability. The following two lemmas make these ideas precise.

Lemma 5.7. *Suppose that* $x, \overline{x} \in c(a^n)$ *and*

$$\overline{d}(\nu_x^n, \nu_{\overline{x}}^n) \le \delta^2.$$

This is the case, for example, if the channel is $\overline{d}$ *continuous and* n *is chosen sufficiently large. Then*

$$\nu_x^n(G_\delta) \ge \nu_{\overline{x}}^n(G) - \delta$$

and hence

$$\inf_{x \in c(a^n)} \nu_x^n(G_\delta) \ge \sup_{x \in c(a^n)} \nu_x^n(G) - \delta.$$

Proof: Again we assume that the infima defining the $\overline{d}$ distance are actually minima and hence there is a pmf p on $B^n \times B^n$ such that

$$\sum_{b^n \in B^n} p(y^n, b^n) = \nu_x^n(y^n)$$

and

$$\sum_{b^n \in B^n} p(b^n, y^n) = \nu_{\overline{x}}^n(y^n);$$

that is, p has ν_x^n and $\nu_{\overline{x}}^n$ as marginals, and

$$\frac{1}{n} E_p d_n(Y^n, \overline{Y}^n) = \overline{d}(\nu_x^n, \nu_{\overline{x}}^n).$$

As previously done, this is true within $\epsilon > 0$ and the proof follows in the same way with inequalities. Using the Markov inequality we can write

$$\begin{aligned}
\nu_x^n(G_\delta) &= p(Y^n \in G_\delta) \ge p(\overline{Y}^n \in G \text{ and } d_n(Y^n, \overline{Y}^n) \le n\delta) \\
&= 1 - p(\overline{Y}^n \notin G \text{ or } d_n(Y^n, \overline{Y}^n) > n\delta) \\
&\ge 1 - p(\overline{Y}^n \notin G) - p(d_n(Y^n, \overline{Y}^n) > n\delta) \\
&\ge \nu_{\overline{x}}^n(G) - \frac{1}{\delta} E(n^{-1} d_n(Y^n, \overline{Y}^n)) \ge \nu_{\overline{x}}^n(G) - \delta
\end{aligned}$$

proving the first statement. The second statement follows from the first. □

Next suppose that $[G, \mu, U]$ is a stationary source, f is a stationary encoder which could correspond to a finite length sliding block encoder or to an infinite length one, ν is a stationary channel, and g is a length m sliding-block decoder. The probability of error for the resulting hookup is defined by

$$P_e(\mu, \nu, f, g) = \Pr(U_0 \neq \hat{U}_0) = \mu\nu(E) = \int d\mu(u)\nu_{f(u)}(E_u),$$

where E is the error event $\{u, y : u_0 \neq g_m(Y_-q^m)\}$ and $E_u = \{y : (u, y) \in E\}$ is the section of E at u.

Lemma 5.8. *Given a stationary channel ν, a stationary source $[G, \mu, U]$, a length m sliding-block decoder, and two encoders f and ϕ, then for any positive integer r*

$$|P_e(\mu, \nu, f, g) - P_e(\mu, \nu, \phi, g)| \leq \frac{m}{r} + r\Pr(f \neq \phi) + m \max_{a^r \in A^r} \sup_{x, x' \in c(a^r)} \overline{d}_r(\nu_x^r, \nu_{x'}^r).$$

Proof: Define $\Lambda = \{u : f(u) = \phi(u)\}$ and

$$\Lambda_r = \{u : f(T^i u) = \phi(T^i u);\ i = 0, 1 \cdots, r-1\} = \bigcap_{i=0}^{r-1} T^i \Lambda.$$

From the union bound

$$\mu(\Lambda_r^c) \leq r\mu(\Lambda^c) = r\Pr(f \neq \phi). \tag{5.45}$$

From stationarity, if $g = g_m(Y_{-q}^m)$ then

$$\begin{aligned} P_e(\mu, \nu, f, g) &= \int d\mu(u)\nu_{f(u)}(y : g_m(y_{-q}^m) \neq u_0) \\ &= \frac{1}{r}\sum_{i=0}^{r-1} \int d\mu(u)\nu_{f(u)}(y : g_m(y_{i-q}^m) \neq u_0) \\ &\leq \frac{m}{r} + \frac{1}{r}\sum_{i=q}^{r-q} \int_{\Lambda_r} d\mu(u)\nu_{f(u)}^r(y^r : g_m(y_{i-q}^m) \neq u_i) + \mu(\Lambda_r^c). \end{aligned} \tag{5.46}$$

Fix $u \in \Lambda_r$ and let p_u yield $\overline{d}_r(\nu_{f(u),\phi(u)}^r)$; that is, $\sum_{w^r} p_u(y^r, w^r) = \nu_{f(u)}^r(y^r)$, $\sum_{y^r} p_u(y^r, w^r) = \nu_{\phi(u)}^r(w^r)$, and

$$\frac{1}{r}\sum_{i=0}^{r-1} p_u(y^r, w^r : y_i \neq w_i) = \overline{d}_r(\nu^r_{f(u),\phi(u)}). \tag{5.47}$$

We have that

$$\begin{aligned}
&\frac{1}{r}\sum_{i=q}^{r-q} \nu^r_{f(u)}(y^r : g_m(y_{i-q}^m) \neq u_i)\\
&= \frac{1}{r}\sum_{i=q}^{r-q} p_u(y^r, w^r : g_m(y_{i-q}^m) \neq u_i)\\
&\le \frac{1}{r}\sum_{i=q}^{r-q} p_u(y^r, w^r : g_m(y_{i-q}^m) \neq w_{i-q}^m) + \frac{1}{r}\sum_{i=q}^{r-q} p_u(y^r, w^r : g_m(w_{i-q}^m) \neq u_i)\\
&\le \frac{1}{r}\sum_{i=q}^{r-q} p_u(y^r, w^r : y_{i-q}^r \neq w_{i-q}^r) + P_e(\mu, \nu, \phi, g)\\
&\le \frac{1}{r}\sum_{i=q}^{r-q}\sum_{j=i-q}^{i-q+m} p_u(y^r, w^r : y_j \neq w_j) + P_e(\mu, \nu, \phi, g)\\
&\le m\overline{d}_r(\nu^r_{f(u)}, \nu^r_{\phi(u)}) + P_e(\mu, \nu, \phi, g),
\end{aligned}$$

which with (5.45)-(5.47) proves the lemma. □

The following corollary states that the probability of error using sliding-block codes over a $\overline{d}$-continuous channel is a continuous function of the encoder as measured by the metric on encoders given by the probability of disagreement of the outputs of two encoders.

Corollary 5.2. *Given a stationary $\overline{d}$-continuous channel ν and a finite length decoder $g_m : B^m \to A$, then given $\epsilon > 0$ there is a $\delta > 0$ so that if f and ϕ are two stationary encoders such that $Pr(f \neq g) \le \delta$, then*

$$|P_e(\mu, \nu, f, g) - P_e(\mu, \nu, \phi, g)| \le \epsilon.$$

Proof: Fix $\epsilon > 0$ and choose r so large that

$$\begin{aligned}
\max_{a^r} \sup_{x,x' \in c(a^r)} \overline{d}_r(\nu_x^r, \nu_{x'}^r) &\le \frac{\epsilon}{3m}\\
\frac{m}{r} &\le \frac{\epsilon}{3},
\end{aligned}$$

and choose $\delta = \epsilon/(3r)$. Then Lemma 5.8 implies that

$$|P_e(\mu, \nu, f, g) - P_e(\mu, \nu, \phi, g)| \le \epsilon.$$

□

Given an arbitrary channel $[A, \nu, B]$, we can define for any block length N a closely related CBI channel $[A, \tilde{\nu}, B]$ as the CBI channel with the same probabilities on output N-blocks, that is, the same conditional probabilities for Y_{kN}^N given x, but having conditionally independent blocks. We shall call $\tilde{\nu}$ the *N-CBI* approximation to ν. A channel ν is said to be *conditionally almost block independent* or *CABI* if given ϵ there is an N_0 such that for any $N \geq N_0$ there is an M_0 such that for any x and any N-CBI approximation $\tilde{\nu}$ to ν

$$\overline{d}(\tilde{\nu}_x^M, \nu_x^M) \leq \epsilon, \text{ all } M \geq M_0,$$

where ν_x^M denotes the restriction of ν_x to $\mathcal{B}_B^N$, that is, the output distribution on Y^N given x. A CABI channel is one such that the output distribution is close (in a $\overline{d}$ sense) to that of the N-CBI approximation provided that N is big enough. CABI channels were introduced by Neuhoff and Shields [133] who provided several examples alternative characterizations of the class. In particular they showed that finite memory channels are both $\overline{d}$-continuous and CABI. Their principal result, however, requires the notion of the $\overline{d}$ distance between channels. Given two channels $[A, \nu, B]$ and $[A, \nu', B]$, define the $\overline{d}$ distance between the channels to be

$$\overline{d}(\nu, \nu') = \limsup_{n \to \infty} \sup_x \overline{d}(\nu_x^n, \nu'^N_x).$$

Neuhoff and Shields [133] showed that the class of CABI channels is exactly the class of primitive channels together with the $\overline{d}$ limits of such channels.

Chapter 6
Distortion and Entropy

Abstract Results are developed relating the goodness of approximation as measured by average Hamming distance between codes and and the d-bar distance between sources to the closeness of entropy rate. A few easy applications provide important properties of entropy rate.

One might suspect that if two codes for a common source closely approximate each other, then the resulting output entropies should also be close. Similarly, it seems reasonable to expect that if two random processes well approximate each other, than their entropy rates should be close. Such notions of continuity of entropy with respect to the goodness of approximation between codes and processes are the focus of this chapter and are fundamental to to the development and extensions to follow. A few easy applications are collected in this chapter.

6.1 The Fano Inequality

A classic result of this type, showing that closeness in the average Hamming distance between two codes forces the entropy to be close, was first proved by Fano and is called Fano's inequality [38]. The result has a variety of extensions and applications.

Lemma 6.1. *Given two finite alphabet measurements f and g on a common probability space $(\Omega, \mathcal{B}, P)$ having a common alphabet A or, equivalently, given the corresponding partitions $\mathcal{Q} = \{f^{-1}(a); a \in A\}$ and $\mathcal{R} = \{g^{-1}(a); a \in A\}$, define the error probability $P_e = |\mathcal{Q} - \mathcal{R}| = Pr(f \neq g)$. Then*

$$H(f|g) \leq h_2(P_e) + P_e \ln(\|A\| - 1)$$

and

$$|H(f) - H(g)| \leq h_2(P_e) + P_e \ln(\|A\| - 1)$$

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and hence entropy is continuous with respect to partition distance for a fixed measure.

Proof: Let $M = \|A\|$ and define a measurement

$$r : A \times A \to \{0, 1, \cdots, M-1\}$$

by $r(a,b) = 0$ if $a = b$ and $r(a,b) = i$ if $a \neq b$ and a is the ith letter in the alphabet $A_b = A - b$. If we know g and we know $r(f,g)$, then clearly we know f since either $f = g$ (if $r(f,g)$ is 0) or, if not, it is equal to the $r(f,g)$th letter in the alphabet A with g removed. Since f can be considered a function of g and $r(f,g)$,

$$H(f|g, r(f,g)) = 0$$

and hence

$$H(f,g,r(f,g)) = H(f|g,r(f,g)) + H(g,r(f,g)) = H(g,r(f,g)).$$

Similarly

$$H(f,g,r(f,g)) = H(f,g).$$

From Lemma 3.2

$$H(f,g) = H(g,r(f,g)) \leq H(g) + H(r(f,g))$$

or

$$\begin{aligned} H(f,g) - H(g) &= H(f|g) \leq H(r(f,g)) \\ &= -P(r=0)\ln P(r=0) - \sum_{i=1}^{M-1} P(r=i)\ln P(r=i). \end{aligned}$$

Since $P(r=0) = 1 - P_e$ and since $\sum_{i\neq 0} P(r=i) = P_e$, this becomes

$$\begin{aligned} H(f|g) &\leq -(1-P_e)\ln(1-P_e) - P_e \sum_{i=1}^{M-1} \frac{P(r=i)}{P_e} \ln \frac{P(r=i)}{P_e} - P_e \ln P_e \\ &\leq h_2(P_e) + P_e \ln(M-1) \end{aligned}$$

since the entropy of a random variable with an alphabet of size $M-1$ is no greater than $\ln(M-1)$. This proves the first inequality. Since $H(f) \leq H(f,g) = H(f|g) + H(g)$, this implies

$$H(f) - H(g) \leq h_2(P_e) + P_e \ln(M-1).$$

Interchanging the roles of f and g completes the proof. □

The lemma can be used to show that related information measures such as mutual information and conditional mutual information are also continuous with respect to the partition metric.

The following corollary extends the the lemma to repeated measurements. Similar extensions may be found in Csiszár and Körner [27].

Again let f and g denote finite-alphabet measurements on a common probability space, but now interpret them as sliding-block codes as in Section 2.6; that is, let T denote a transformation on the common space (e.g., the shift on a sequence space) and define $f_i = fT^i$, $g = gT^i$ so that $\{f_n, g_n\}$ is a pair process with a common finite alphabet. Define the nth order per-symbol or mean probability of error

$$P_e^{(n)} = \frac{1}{n} \sum_{i=0}^{n-1} \Pr(f_i \neq g_i)$$

and observe that this is simply the normalized average of the additive fidelity criterion corresponding to the Hamming distance:

$$P_e^{(n)} = \frac{1}{n} E\left(\sum_{i=0}^{n-1} d_H(x_i, y_i) \right).$$

If the transformation (or the pair process) is stationary, then $P_e^{(n)} = P_e^{(1)} = P_e$.

Corollary 6.1. *Given two sequences of measurements $\{f_n\}$ and $\{g_n\}$ with finite alphabet A on a common probability space,*

$$\frac{1}{n} H(f^n|g^n) \leq P_e^{(n)} \ln(\|A\| - 1) + h_2(P_e^{(n)})$$

and

$$|\frac{1}{n} H(f^n) - \frac{1}{n} H(g^n)| \leq P_e^{(n)} \ln(\|A\| - 1) + h_2(P_e^{(n)}).$$

If $\{f_n, g_n\}$ are also AMS and hence the limit

$$\overline{P}_e = \lim_{n \to \infty} P_e^{(n)}$$

exists, then if we define

$$\overline{H}(f \mid g) = \lim_{n \to \infty} \frac{1}{n} H(f^n \mid g^n) = \lim_{n \to \infty} \frac{1}{n} (H(f^n, g^n) - H(g^n)),$$

where the limits exist since the processes are AMS, then

$$\overline{H}(f|g) \leq \overline{P}_e \ln(\|A\| - 1) + h_2(\overline{P}_e)$$
$$|\overline{H}(f) - \overline{H}(g)| \leq \overline{P}_e \ln(\|A\| - 1) + h_2(\overline{P}_e).$$

Proof: From the chain rule for entropy (Corollary 3.6), Lemma 3.12, and Lemma 6.1

$$H(f^n|g^n) = \sum_{i=0}^{n-1} H(f_i|f^i, g^n) \le \sum_{i=0}^{n-1} H(f_i|g^i) \le \sum_{i=0}^{n-1} H(f_i|g_i)$$
$$\le \sum_{i=0}^{n-1} \left(\Pr(f_i \ne g_i) \ln(\|A\| - 1) + h_2(\Pr(f_i \ne g_i))\right)$$

from the previous lemma. Dividing by n yields the first inequality which implies the second as in the proof of the previous lemma. If the processes are jointly AMS, then the limits exist and the entropy rate results follows from the continuity of h_2 by taking the limit. □

6.2 Code Approximation and Entropy Rate

Corollary 6.1 has two simple but extremely important implications summarized in the next corollary. The first part is immediate, that two sliding-block codes which closely approximate each other in the code distance must have approximately the same entropy rate. The second part applies this observation to draw the similar conclusion that entropy rate of a source is a continuous function with respect to the Ornstein d-bar distance.

Corollary 6.2. *For a fixed AMS source, entropy rate of a sliding-block encoding of the source is a continuous function of the (Hamming) code distance.*

Entropy rate is a continuous function of the source with respect to the Ornstein d-bar distance.

Proof. The first part follows since $\overline{P}_e$ in Corollary 6.1 is the code distance between sliding-block codes f and g.

For the second part, if we can make $\overline{d}_H$ between two processes $\{f_n\}$ and $\{g_n\}$ arbitrarily small, then there is a coupling which yields an average Hamming distortion $\overline{P}_e$ as small as we would like, which in turn implies from Corollary 6.1 that the two entropy rates are small. □

Combining Lemma 5.2 and Corollary 6.1 immediately yields the following corollary, which permits us to study the entropy rate of general stationary codes by considering codes which depend on only a finite number of inputs (and hence for which the ordinary entropy results for random vectors can be applied).

Corollary 6.3. *Let f be a stationary code of an AMS process X. As in (5.24–5.25) define for positive integers n define $\mathcal{F}_n = \sigma(X_0, X_1, \cdots, X_n)$*

in the one-sided case and $\sigma(X_{-n}, \cdots, X_n)$ *in the two-sided case. Given* $\epsilon > 0$ *there exists for sufficiently large* n *a code* g *measurable with respect to* $\mathcal{F}_n$ *such that*

$$| \overline{H}(f) - \overline{H}(g) | \leq \epsilon.$$

Corollary 6.3 can be used to show that entropy rate, like entropy, is reduced by coding. The general stationary code is approximated by a code depending on only a finite number of inputs and then the result that entropy is reduced by mapping (Lemma 3.3) is applied.

Corollary 6.4. *Given an AMS process* $\{X_n\}$ *and a stationary coding* f *of the process, then*

$$\overline{H}(X) \geq \overline{H}(f),$$

that is, stationary coding reduces entropy rate.

Proof: For integer n define $\mathcal{F}_n = \sigma(X_0, X_1, \cdots, X_n)$ in the one-sided case and $\sigma(X_{-n}, \cdots, X_n)$ in the two-sided case. Then $\mathcal{F}_n$ asymptotically generates $\mathcal{B}(A_X)^\infty$. Hence given a code f and an $\epsilon > 0$ we can choose using the finite alphabet special case of the previous lemma a large k and a $\mathcal{F}_k$-measurable code g such that $|\overline{H}(f) - \overline{H}(g)| \leq \epsilon$. We shall show that $\overline{H}(g) \leq \overline{H}(X)$, which will prove the lemma. To see this in the one-sided case observe that g is a function of X^k and hence g^n depends only on X^{n+k} and hence

$$H(g^n) \leq H(X^{n+k})$$

and hence

$$\overline{H}(g) = \lim_{n\to\infty} \frac{1}{n} H(g^n) \leq \lim_{n\to\infty} \frac{1}{n} \frac{n}{n+k} H(X^{n+k}) = \overline{H}(X).$$

In the two-sided case g depends on $\{X_{-k}, \cdots, X_k\}$ and hence g_n depends on $\{X_{-k}, \cdots, X_{n+k}\}$ and hence

$$H(g^n) \leq H(X_{-k}, \cdots, X_{-1}, X_0, \cdots, X_{n+k}) \leq H(X_{-k}, \cdots, X_{-1}) + H(X^{n+k}).$$

Dividing by n and taking the limit completes the proof as before. □

Dynamical Systems and Random Processes

It is instructive to apply Corollary 6.4 to relate the idea of the entropy of a dynamical system with the entropy rate of a random process. The result is not required for later coding theorems, but it provides insight into the connections between entropy as considered in ergodic theory and entropy as used in information theory. In addition, the development involves some ideas of coding and approximation which are useful in

proving the ergodic theorems of information theory used to prove coding theorems.

Let $\{X_n\}$ be a random process with alphabet A_X. Let A_X^∞ denote the one or two-sided sequence space. Consider the dynamical system $(\Omega, \mathcal{B}, P, T)$ defined by $(A_X^\infty, \mathcal{B}(A_X)^\infty, P, T)$, where P is the process distribution and T the shift. Recall from Section 3.1 and Section 2.6 that a stationary coding or infinite length sliding-block coding of $\{X_n\}$ is a measurable mapping $f : A_X^\infty \to A_f$ into a finite alphabet which produces an encoded process $\{f_n\}$ defined by

$$f_n(x) = f(T^n x);\ x \in A_X^\infty.$$

The entropy $H(P,T)$ of the dynamical system was defined in (3.4) by

$$H(P,T) = \sup_f \overline{H}_P(f),$$

the supremum of the entropy rates of finite alphabet stationary codings of the original process. We shall show that if the original alphabet is finite, then the entropy of the dynamical system is exactly the entropy rate of the process.

Theorem 6.1. *Let $\{X_n\}$ be a random process with finite alphabet A_X. Let A_X^∞ denote the one or two-sided sequence space. Consider a dynamical system $(\Omega, \mathcal{B}, P, T)$ defined by $(A_X^\infty, \mathcal{B}(A_X)^\infty, P, T)$, where P is an AMS process distribution and T is the shift. Then*

$$H(P,T) = \overline{H}(X).$$

Proof: From (3.5), $H(P,T) \geq \overline{H}(X)$. Conversely suppose that f is a code which yields $\overline{H}(f) \geq H(P,T) - \epsilon$. Since f is a stationary coding of the AMS process $\{X_n\}$, Corollary 6.4 implies that $\overline{H}(f) \leq \overline{H}(X)$. Thus $H(P,T) - \epsilon \leq \overline{H}(X)$, which completes the proof since ϵ is arbitrary. □

6.3 Pinsker's and Marton's Inequalities

Fano's inequality shows that small probability of error between two processes implies that the entropy rates of the processes must be close. A converse of sorts for the special case where one of the two processes is IID follows from an inequality of Marton, which in turn follows from an inequality of Pinsker. In this section these inequalities are derived and discussed.

The following inequality provides an upper bound to the variation distance between to probability measures in terms of the divergence (see [151], p. 58 of [27], (13) of [166]).

Lemma 6.2. Pinsker's Inequality

Given two probability measures M and P, the variation distance and divergence satisfy the inequality

$$\mathrm{var}(P, M) \le \sqrt{2D(P\|M)}.$$

Proof. Assume that $M \gg P$ since the result is trivial otherwise because the right-hand side is infinite. The inequality will follow from the first statement of Lemma 5.4 and the following inequality: Given $1 \ge p, m \ge 0$,

$$p \ln \frac{p}{m} + (1-p) \ln \frac{1-p}{1-m} - 2(p-m)^2 \ge 0. \tag{6.1}$$

To see this, suppose the truth of (6.1). Since F can be chosen so that $2(P(F) - M(F))$ is arbitrarily close to $d(P, M)$, given $\epsilon > 0$ choose a set F such that $[2(P(F) - M(F))]^2 \ge d(P, M)^2 - 2\epsilon$. Since $\{F, F^c\}$ is a partition,

$$D(P\|M) - \frac{d(P,M)^2}{2} \ge$$
$$P(F) \ln \frac{P(F)}{M(F)} + (1 - P(F)) \ln \frac{1-P(F)}{1-M(F)} - 2(P(F) - M(F))^2 - \epsilon.$$

If (6.1) holds, then the right-hand side is bounded below by $-\epsilon$, which proves the lemma since ϵ is arbitrarily small. To prove (6.1) observe that the left-hand side equals zero for $p = m$, has a negative derivative with respect to m for $m < p$, and has a positive derivative with respect to m for $m > p$. (The derivative with respect to m is $(m-p)[1 - 4m(1-m)]/[m(1-m)]$.) Thus the left hand side of (6.1) decreases to its minimum value of 0 as m tends to p from above or below. □

The lemma together with Lemma 5.5 yield the following corollary.

Corollary 6.5. *Given two probability measures M and P, the transportation with respect to the Hamming distance (the first order Ornstein's d-bar distance) and divergence satisfy the inequality*

$$\overline{d}_H(P, M) \le \sqrt{\frac{D(P\|M)}{2}}.$$

Marton extended Pinsker's inequality to vectors produced by processes when one of the processes is memoryless and thereby obtained in the limit an inequality between Ornstein's d-bar distance between processes and the relative entropy rate of an arbitrary process with respect to the IID process [120]. She subsequently extended this result from IID to a class of Markov processes [121], but we shall concentrate on the IID result, which is stated in the following lemma. The proof follows Shields [166].

Lemma 6.3. Marton's Inequality *Suppose that X^n and Y^n are random vectors with a common finite alphabet and probability mass functions μ_{X^n} and μ_{Y^n} and that Y^n is memoryless (μ_{Y^n} is a product pmf). Then the d-bar distance (mean Hamming) satisfies*

$$\frac{1}{n}\overline{d}_n(\mu_{X^n}, \mu_{Y^n}) \le \sqrt{\frac{D(\mu_{X^n} \| \mu_{Y^n})}{2n}}. \tag{6.2}$$

and hence in the limit

$$\overline{d}(\mu_X, \mu_Y) \le \sqrt{\frac{\overline{D}(\mu_X \| \mu_Y)}{2}} \tag{6.3}$$

Proof. Suppose that the common finite alphabet for X_n and Y_n is A. First note that for the case $n = 1$ Marton's inequality and Pinker's inequality are the same so that

$$\overline{d}_1(\mu_{X_0}, \mu_{Y_0}) \le \sqrt{\frac{D(\mu_{X_0} \| \mu_{Y_0})}{2}}. \tag{6.4}$$

Denote by $p^{(1)}(x_0, y_0)$ the coupling yielding (with the usual caveat) the Hamming transportation distance $\overline{d}_1(\mu_{X_0}, \mu_{Y_0})$, that is, the pmf on $A \times A$ with marginals μ_{X_0} and μ_{Y_0} yielding the smallest $E(d_H(X_0, Y_0)) = \Pr(X_0 \ne Y_0)$.

For $n \ge 2$ consider Pinsker's inequality applied to the distributions P and M corresponding to the probability mass functions $\mu_{X_{n-1}|X^{n-1}}(x_{n-1} \mid x^{n-1})$ and $\mu_{Y_{n-1}}(y_{n-1})$:

$$\overline{d}_{\mathrm{H}}(\mu_{X_{n-1}|X^{n-1}}(\cdot \mid x^{n-1}), \mu_{Y_{n-1}}) \le \sqrt{\frac{D(\mu_{X_{n-1}|X^{n-1}}(\cdot \mid x^{n-1}) \| \mu_{Y_{n-1}})}{2}}, x^n \in A^n. \tag{6.5}$$

Let $p^{(n)}(x_{n-1}, y_{n-1} \mid x^{n-1})$ denote the coupling (on $A \times A$) between the pmfs $\mu_{X_{n-1}|X^{n-1}}(\cdot \mid x^{n-1})$ and $\mu_{Y_{n-1}}$ on A yielding $\overline{d}_{\mathrm{H}}(\mu_{X_{n-1}|X^{n-1}}(\cdot \mid x^{n-1}), \mu_{Y_{n-1}})$ (these exist from Theorem 5.2).

Taking expectations in (6.5) yields

$$\sum_{x^{n-1}} \mu_{X^{n-1}}(x^{n-1}) \overline{d}_{\mathrm{H}}(\mu_{X_{n-1}|X^{n-1}}(\cdot \mid x^{n-1}), \mu_{Y_{n-1}}) \le \sum_{x^{n-1}} \mu_{X^{n-1}}(x^{n-1}) \sqrt{\frac{D(\mu_{X_{n-1}|X^{n-1}}(\cdot \mid x^{n-1}) \| \mu_{Y_{n-1}})}{2}}. \tag{6.6}$$

Use these pmfs to construct a new pmf $\pi^{(n)}(x^n, y^n)$ on $A^n \times A^n$ defined by

$$\pi^{(n)}(x^n, y^n) = p^{(1)}(x_0, y_0) \prod_{i=2}^{n} p^{(i)}(x_{i-1}, y_{i-1} \mid x^{i-1}).$$

This joint pmf has as its marginals μ_{X^n} and μ_{Y^n} and hence is a coupling of these two distributions. This implies that

$$E_{\pi^{(n)}}(d_n(X^n, Y^n)) \geq \overline{d}_n(\mu_{X^n}, \mu_{Y^n}). \tag{6.7}$$

We also have that

$$\begin{aligned}
&E_{\pi^{(n)}}(d_n(X^n, Y^n)) \\
&= E_{\pi^{(n)}}\left(\sum_{i=0}^{n-1} d_H(X_i, Y_i)\right) = \sum_{i=0}^{n-1} E_{\pi^{(n)}}(d_H(X_i, Y_i)) = \sum_{i=0}^{n-1} E_{\pi^{(i)}}(d_H(X_i, Y_i)) \\
&= \sum_{x_0}\sum_{y_0} p^{(1)}(x_0, y_0) d_H(X_0, Y_0) \\
&\quad + \sum_{i=2}^{n} \sum_{x^{i-1}} \mu_{X^{i-1}}(x^{i-1}) \sum_{x_{i-1}, y_{i-1}} p^{(i)}(x_{i-1}, y_{i-1} \mid x^{i-1}) d_H(x_{i-1}, y_{i-1}) \\
&= \overline{d}_H(\mu_{X_0}, \mu_{Y_0}) + \sum_{i=2}^{n} \sum_{x^{i-1}} \mu_{X^{i-1}}(x^{i-1}) \overline{d}_H(\mu_{X_{i-1}|X^{i-1}}(\cdot \mid x^{i-1}), Y_{i-1}).
\end{aligned}$$

Apply Pinsker's inequality from (6.4) and (6.6) and use (6.7) to write

$$\begin{aligned}
&\overline{d}_n(\mu_{X^n}, \mu_{Y^n}) \\
&\leq \overline{d}_H(\mu_{X_0}, \mu_{Y_0}) + \sum_{i=2}^{n} \sum_{x^{i-1}} \mu_{X^{i-1}}(x^{i-1}) \overline{d}_H(\mu_{X_{i-1}|X^{i-1}}(\cdot \mid x^{i-1}), Y_{i-1}) \\
&\leq \sqrt{\frac{D(\mu_{X_0} \| \mu_{Y_0})}{2}} + \sum_{i=2}^{n} \sum_{x^{i-1}} \mu_{X^{i-1}}(x^{i-1}) \sqrt{\frac{D(\mu_{X_{i-1}|X^{i-1}}(\cdot \mid x^{i-1}) \| \mu_{Y_{i-1}})}{2}}.
\end{aligned}$$

Use the concavity of the square root twice to obtain

$$\begin{aligned}
\frac{\overline{d}_n(\mu_{X^n}, \mu_{Y^n})}{n} &\leq \frac{1}{n}\sqrt{\frac{D(\mu_{X_0} \| \mu_{Y_0})}{2}} + \\
&\quad \frac{1}{n}\sum_{i=2}^{n} \sqrt{\frac{\sum_{x^{i-1}} \mu_{X^{i-1}}(x^{i-1}) D(\mu_{X_{i-1}|X^{i-1}}(\cdot \mid x^{i-1}) \| \mu_{Y_{i-1}})}{2}} \\
&\leq \sqrt{\frac{1}{2n}\left(D(\mu_{X_0} \| \mu_{Y_0}) + \sum_{i=2}^{n} \sum_{x^{i-1}} \mu_{X^{i-1}}(x^{i-1}) D(\mu_{X_{i-1}|X^{i-1}}(\cdot \mid x^{i-1}) \| \mu_{Y_{i-1}})\right)}.
\end{aligned}$$

The following string of equalities shows that the term in parentheses is $D(\mu_{X^n} \| \mu_{Y^n})$, which completes the proof of (6.2):

$$
\begin{aligned}
&D(\mu_{X^n}\|\mu_{Y^n}) \\
&= \sum_{x^n} \mu_{X^n}(x^n) \ln \frac{\mu_{X^n}(x^n)}{\mu_{Y^n}(x^n)} \\
&= \sum_{x^n} \mu_{X^n}(x^n) \ln \frac{\mu_{X_0}(x_0)\prod_{i=1}^{n-1}\mu_{X_i|X^i}(x_i \mid x^i)}{\mu_{Y_0}(x_0)\prod_{i=1}^{n-1}\mu_{Y_i}(y_i)} \\
&= \sum_{x^n} \mu_{X^n}(x^n) \ln \left(\frac{\mu_{X_0}(x_0)}{\mu_{Y_0}(x_0)} \prod_{i=1}^{n-1} \frac{\mu_{X_i|X^i}(x_i \mid x^i)}{\mu_{Y_i}(y_i)} \right) \\
&= \sum_{x^n} \mu_{X^n}(x^n) \ln \frac{\mu_{X_0}(x_0)}{\mu_{Y_0}(x_0)} + \sum_{i=1}^{n-1} \sum_{x^n} \mu_{X^n}(x^n) \ln \frac{\mu_{X_i|X^i}(x_i \mid x^i)}{\mu_{Y_i}(y_i)} \\
&= D(\mu_{X_0}\|\mu_{Y_0}) + \sum_{i=1}^{n-1} \sum_{x^i} \mu_{X^i}(x^i) \sum_{x_i} \mu_{X_i|X^i}(x_i \mid x^i) \ln \frac{\mu_{X_i|X^i}(x_i \mid x^i)}{\mu_{Y_i}(y_i)} \\
&= D(\mu_{X_0}\|\mu_{Y_0}) + \sum_{i=1}^{n-1} \sum_{x^i} \mu_{X^i}(x^i) D(\mu_{X_i|X^i}(\cdot \mid x^i)\|\mu_{Y_i}).
\end{aligned}
$$

□

While Fano's inequality deals with conditional entropy at its most basic level, Marton's inequality deals with relative entropy. Just as Fano's inequality results in a relationship between the d-bar distance and the difference of entropy rates, Marton's inequality also has an implication for entropy rates. If Y is IID and equiprobable as in the case of fair coin flips, Marton's inequality immediately yields the following corollary.

Corollary 6.6. *Suppose that μ_X and μ_Y are distributions of two stationary processes with a common finite alphabet and that Y is both IID and has equiprobable marginals. Then*

$$\overline{d}(X,Y) \le \sqrt{\frac{\overline{H}(Y) - \overline{H}(X)}{2}}.$$

Thus if an arbitrary stationary process has entropy rate close to that of an IID equiprobable source with the same finite alphabet, then it must be also close in Ornstein's d-bar distance.

6.4 Entropy and Isomorphism

The results derived thus far in this chapter have as an easy application one of the most important results of ergodic theory, the Kolmogorov-Sinai theorem demonstrating that a necessary condition for two dynamical systems to be isomorphic is that they have the same entropy rate.

Roughly speaking, two random processes are isomorphic if each can be coded into the other in a stationary and invertible way. The primary difficulty in making this result precise is developing the necessary definitions, which will be related to the coding language used here. The focus is on dynamical systems rather than on random processes because the latter are more general and form the traditional context for treating isomorphic processes. The initial material follows [58].

There are several notions of isomorphism: isomorphic measurable spaces, isomorphic probability spaces, and isomorphic dynamical systems. Isomorphic random processes are a special case of the latter.

Isomorphic Measurable Spaces

Two measurable spaces $(\Omega, \mathcal{B})$ and (Λ, S) are *isomorphic* if there exists a measurable function $\phi : \Omega \to \Lambda$ that is one-to-one and has a measurable inverse ϕ^{-1}. In other words, the inverse image $\phi^{-1}(\lambda)$ of a point $\lambda \in \Lambda$ consists of exactly one point in Ω and the inverse mapping so defined, say $\gamma : \Lambda \to \Omega$, $\gamma(\lambda) = \phi^{-1}(\lambda)$, is itself a measurable mapping. The function ϕ (or its inverse γ) with these properties is called an *isomorphism.* An isomorphism between two measurable spaces is thus an invertible mapping between the two sample spaces that is measurable in both directions.

Isomorphic Probability Spaces

Two probability spaces $(\Omega, \mathcal{B}, P)$ and (Λ, S, Q) are *isomorphic* if there is an isomorphism $\phi : \Omega \to \Lambda$ between the two measurable spaces $(\Omega, \mathcal{B})$ and (Λ, S) with the added property that

$$Q = P_\phi \text{ and } P = Q_{\phi^{-1}}$$

that is,

$$Q(F) = P(\phi^{-1}(F)); F \in S; P(G) = Q(\phi(G)); G \in \mathcal{B}.$$

Two probability spaces are isomorphic

1. if one can find for each space a random variable defined on that space that has the other as its output space, and
2. the random variables can be chosen to be inverses of each other; that is, if the two random variables are ϕ and γ, then $\phi(\gamma(\lambda)) = \lambda$ and $\gamma(\phi(\omega)) = \omega$.

Note that if the two probability spaces $(\Omega, \mathcal{B}, P)$ and (Λ, S, Q) are isomorphic and $\phi : \Omega \to \Lambda$ is an isomorphism with inverse γ, then the random variable $\phi\gamma$ defined by $\phi\gamma(\lambda) = \phi(\gamma(\lambda))$ is equivalent to the identity random variable $i : \Lambda \to \Lambda$ defined by $i(\lambda) = \lambda$.

Isomorphism Mod 0

A weaker notion of isomorphism between two probability spaces is that of *isomorphism mod 0* . Two probability spaces are isomorphic mod 0 or *metrically isomorphic* if the mappings have the desired properties on sets of probability 1, that is, the mappings can be defined except for null sets in the respective spaces. Thus two probability spaces $(\Omega, \mathcal{B}, P)$ and (Λ, S, Q) are isomorphic (mod 0) if there are null sets $\Omega_0 \in \mathcal{B}$ and $\lambda_0 \in S$ and a measurable one-to-one onto map $\phi : \Omega - \Omega_0 \to \Lambda - \Lambda_0$ with measurable inverse such that $Q(F) = P(\phi^{-1}(F))$ for all $F \in S$. This weaker notion is the standard one in ergodic theory.

Isomorphic Dynamical Systems

Roughly speaking, two dynamical systems are isomorphic if one can be coded or filtered onto the other in an invertible way so that the coding carries one transformation into the other, that is, one can code from one system into the other and back again and coding and transformations commute.

Two dynamical systems $(\Omega, \mathcal{B}, P, S)$ and (Λ, S, m, T) are *isomorphic* if there exists an isomorphism $f : \Omega \to \Lambda$ such that $T\phi(\omega) = \phi(S\omega); \omega \in \Omega$. As with the isomorphism of probability spaces, isomorphic mod 0 means that the properties need hold only on sets of probability 1, but we also require that the null sets in the respective spaces on which the isomorphism is not defined to be invariant with respect to the appropriate transformation. Henceforth isomorphism of dynamical systems will be taken to mean isomorphism mod 0 with this constraint.

Isomorphic Random Processes

Suppose that the probability space (Λ, S, m) is the sequence space of a directly given finite-alphabet random process, say $(A_X^{\mathbb{T}}, \mathcal{B}_{A_X}^{\mathbb{T}}, \mu_X)$, T is the shift on this space, and Π_0 the sampling function on this space, then the random process $X_n = \Pi_0 T^n$ defined on (Λ, S, m) is equivalent to the ran-

dom process $\Pi_0(\phi S^n)$ defined on the probability space $(\Omega, \mathcal{B}, P)$. More generally, any random process of the form gT^n defined on (Λ, S, m) is equivalent to the random process $g(\phi S^n)$ defined on the probability space $(\Omega, \mathcal{B}, P)$. A similar conclusion holds in the opposite direction. Thus, any random process that can be defined on one dynamical system as a function of transformed points possesses an equivalent model in terms of the other dynamical system and its transformation. In addition, not only can one code from one system into the other, one can recover the original sample point by inverting the code (at least with probability 1).

Isomorphism provides a variety of equivalent models for random processes. The models can be quite different in appearance, yet each can be transformed into the other by coding (for discrete alphabet processes) or filtering (for continuous alphabet processes).

If $X = \{X_n\}$ and $Y = \{Y_n\}$ are two finite-alphabet random processes and they are isomorphic, then each is equivalent to a random process formed by a stationary or sliding-block coding of the other. Equivalent processes have the same process distributions and hence the same entropy rates. Suppose that Y is equivalent to a stationary coding f of X and that X is equivalent to a stationary coding $g = f^{-1}$ of Y. From Corollary 6.3, $\overline{H}(X) \geq \overline{H}(f(X)) = \overline{H}(Y)$ and $\overline{H}(Y) \geq \overline{H}(g(X)) = \overline{H}(X)$ and hence $\overline{H}(X) = \overline{H}(Y)$. This yields the random process special case of one of the most famous results of ergodic theory, originally due to Kolmogorov and Sinai. It is summarized in the following theorem.

Theorem 6.2. Kolmogorov-Sinai Theorem *A necessary condition for two AMS random processes to be isomorphic is that they have the same entropy rates.*

In the 1970s, Donald Ornstein proved in a remarkable series of papers that were summarized in [139, 140] that equal entropy was also a sufficient condition for two processes to be isomorphic if the processes were *B*-processes, stationary codings or filterings of IID processes. This result is now known as Ornstein's isomorphism theorem or as the Kolmogorov-Sinai-Ornstein isomorphism theorem. The general result is beyond the scope of this book and no attempt at a proof will be made here. On the other hand, stating the result reinforces the importance of the concept of entropy and entropy rate beyond the domain of information and coding theory, and the result is useful for obtaining and interpreting results relating source coding and simulating random processes, results which have not yet yielded to simpler proofs. For reference we state without proof the Ornstein theorem and a related result due to Sinai which is used in the proof of Ornstein's theorem. Excellent accessible (with respect to the original papers) treatments of the Ornstein theory and the Sinai theorem can be found in books by Shields [164] and by Kalikow and McCutcheon [84].

Theorem 6.3. Ornstein Isomorphism Theorem *Two B-processes are isomorphic if and only if they have the same entropy rate.*

Theorem 6.4. Sinai's Theorem *Suppose that $\{U_n\}$ is a stationary and ergodic process with entropy rate $\overline{H}(U)$, that $\{X_n\}$ is a B-process with distribution μ_X and entropy rate $\overline{H}(X)$, and that $H(X) \le H(U)$. Then there is a sliding-block coding of U that has distribution μ_X.*

Sinai's theorem implies half the Ornstein theorem by showing that a specified B-process can always be obtained by stationary coding of *any* stationary and ergodic process having equal or greater entropy rate. The hard part, however, is the other half — the demonstration that if the entropy rates are equal and the process being encoded is also required to be a B-process, then the stationary code can be made invertible. The usual statement of the Sinai theorem is less general and considers the case where $\{X_n\}$ is IID rather than a B-process. The more general result quoted here can be found, e.g., as 636 Theorem in [84], p. 141.

6.5 Almost Lossless Source Coding

The Ornstein and Sinai theorems at the heart of modern ergodic theory are difficult to prove and require intricate and long arguments. Source coding ideas using the properties of entropy and the Rohlin theorem, however, yield approximate versions of the Ornstein results with proofs that are germane to the present development and provide a relatively simple example of techniques to be used in proving coding theorems in later chapters. The results also provide useful interpretations of the Ornstein and Sinai results as idealized source coding and simulation, and of almost lossless source coding as an approximation to isomorphism and the Sinai theorem. To simplify the notation and the presentation, this section concentrates on an archetypal problem of almost lossless source coding, that is, of converting a stationary and ergodic discrete-alphabet source with an arbitrary alphabet and known entropy rate into bits in a way that is *almost lossless* in that the coding allows recovery of the original sequence with small probability of error. This is a special case coding for small average distortion with respect to a Hamming distance on symbols. For simplicity and clarity, for the time being only the special case of a source with entropy rate of 1 bit per symbol is considered. The goal is to characterize the behavior of sliding-block codes. More general results will be developed later.

Suppose that $\{X_n\}$ is a stationary and ergodic source with a discrete alphabet A, process distribution μ_X, and entropy rate $\overline{H}(X) = \overline{H}(\mu_X) = 1$, where base 2 logarithms will be used throughout this section.

Almost-Lossless Block Codes

The first step is a slight variation of the block code constructed using the asymptotic equipartition property (AEP) of Section 4.5. Given $\epsilon > 0$, there is an n_0 sufficiently large so that for all $n \geq n_0$ the set of entropy-typical sequences

$$G_n = \{x^n : 2^{-n(\overline{H}(X)+\epsilon)} \leq \mu_{X^n}(x^n) \leq 2^{-n(\overline{H}(X)-\epsilon)}\}$$

satisfies

$$\mu_{X^n}(G_n^c) \leq \epsilon.$$

As outlined in Section 4.5, an initial approach to coding source n tuples into binary n-tuples is to index the length n sequences in G_n by binary n-tuples, encode each input vector into the index if the vector is in G_n and some arbitrary binary n-tuple otherwise, and then decode the binary n-tuple into the corresponding x^n. With high probability (greater than $1 - \epsilon$) the decoded vector will be the input vector, which in turn implies the symbols will be correct with high probability. An immediate problem, however, is that in general there will be too many vectors in G_n. Observe that the number of vectors G_n, $\|G_n\|$, can be bound above using the inequality

$$\begin{aligned} 1 \geq \Pr(X^n \in G_n) &= \sum_{x^n \in G_n} \mu_{X^n}(x^n) \\ &\geq \sum_{x^n \in G_n} 2^{-n(\overline{H}(X)+\epsilon)} = \|G_n\| 2^{-n(1+\epsilon)} \end{aligned}$$

so that $\|G_n\| \leq 2^{n(1+\epsilon)}$, but there are only 2^n available binary n-tuples as indices. Since there can be more than 2^n entropy-typical sequences of a source with entropy rate 1, there can be too few indices in $\{0,1\}^n$ for G_n. We could avoid this problem by requiring that the source have entropy rate $\overline{H}(X)$ strictly less than 1, but it is desirable to consider the case where the source has the maximal entropy rate that can be squeezed through the binary encoded process, which is the 1 bit per sample of fair coin flips. So instead we modify the scheme.

One possible modification is as follows. We will use a slightly larger blocklength N for the code than the vector dimension n of the entropy-typical vectors. Toward this end choose $\delta_0 > 0$ and set

$$k = k(N) = \lfloor \delta_0 N \rfloor + 1 = \lceil \delta_0 N \rceil, \tag{6.8}$$

where as usual $\lfloor r \rfloor$ is the greatest integer less than or equal to r. and $n = N - k$ so that $N = n + k$. Note for later use that

$$\delta_0 N \le k \le \delta_0 N + 1 \tag{6.9}$$

$$(1-\delta_0)N - 1 \le n \le N(1-\delta_0). \tag{6.10}$$

The encoder ignores the first k symbols of each input block. If the n-tuple following the first k symbols is in G_n, it will be coded into the binary N-tuple index. Map all n-tuples not in G_n into a fixed binary n-tuple, say the all 0 n-tuple. We need to reserve $2^{n(1+\epsilon)}$ indices as above, and we have 2^N binary N-tuples, so there will be enough indices for all of the entropy-typical N-tuples if $(N-k)(1+\epsilon) \le N$ or

$$\frac{N\epsilon}{1+\epsilon} \le k = \lceil \delta_0 N \rceil.$$

This will be true if $\delta_0 \ge \epsilon/(1+\epsilon)$, so choose

$$\delta_0 = \epsilon.$$

The decoder will map the first $k \approx \epsilon N$ symbols of each block into an arbitrary k-tuple such as an all 0 k-tuple. The remaining $n \approx N(1-\epsilon)$ binary symbols will be viewed as an index into G_n and the vector indexed by the binary n-tuple will be the output. Note that if the corresponding input n-tuple was in fact in G_n, all of the n corresponding output symbols will be decoded correctly. Let $\hat{X}_i$ denote the resulting reconstruction symbols. The average error probability over a block satisfies

$$P_e^{(N)} = \frac{1}{N}\sum_{i=0}^{N-1} \Pr(X_i \ne \hat{X}_i) \le \frac{k}{N} + \frac{n}{N}\frac{1}{n}\sum_{i=k}^{k+n-1} \Pr(X_i \ne \hat{X}_i).$$

Since the event $\{X_i \ne \hat{X}_i\}$ for some $i \in \{k, k+1, \ldots, k+n-1\}$ is a subset of the event $\{X_k^n \ne \hat{X}_k^n\}$, $\Pr(X_i \ne \hat{X}_i) \le \Pr(X_k^n \notin G_n)$ so that with stationarity,

$$P_e^{(N)} \le \frac{k}{N} + \frac{n}{N}\mu_{X^n}(G_n^c) \le \epsilon + \frac{1}{N} + \mu_{X^n}(G_n^c), \tag{6.11}$$

where we have used (6.9). Invoking the AEP of Section 4.5 with N satisfying $N(1-\delta_0) \ge n_0$ we have

$$P_e^{(N)} \le 2\epsilon + \frac{1}{N}. \tag{6.12}$$

Thus there is an N_0 such that

$$P_e^{(N)} \le 3\epsilon, \text{ all } N \ge N_0.$$

so the average probability of error can be made as small as we would like by choosing a sufficiently large blocklength for the block code. Note that

$$P_e^{(N)} = \frac{1}{N} \sum_{i=0}^{N-1} E\left(d_H(X_i, \hat{X}_i)\right),$$

the average Hamming distortion for the block.

In the limit of large block length, block codes can be used to achieve small average distortion in the sense that a discrete alphabet source with entropy rate 1 bit per symbol can be coded into a binary sequence from which we can recover the original source with asymptotically vanishing mean per symbol error probability. As previously discussed, however, a problem with the reproduction sequence $\hat{X}_n$ in a block coding system is that it will be neither stationary nor ergodic in general, and hence the reconstructed sequence lacks important statistical properties of the original source. In practical terms, there might (and often will) be artifacts in the reproduction due to the blocking, and these artifacts can be objectionable perceptually even if the average distortion is small. The end goal of this section will be to construct a sliding-block code with similar average distortion, but having the property that the reproduction (and the binary encoded sequence) is both stationary and ergodic.

Asynchronous Block Code

Essential to the operation of an ordinary block code is the synchronization between the decoder and encoder — the decoder knows a priori where the code blocks begin so it knows how to interpret binary N-tuples as indices. When the block code is stationarized to form a sliding-block code, this synchronization is lost. As the next step towards constructing a sliding-block code we again modify the block code so that the binary codewords can be located even if the decoder does not know a priori where the block boundaries are. Before synchronizing the code, assume that ϵ is fixed as before and that the encoder blocks are divided as before into an initial $k \approx N\epsilon$ symbols which will be ignored, followed by $n \approx N(1-\epsilon)$ source symbols to be block coded using G_n. Now focus on the binary index codebook containing a subset of $\{0,1\}^N$ and on the decoder.

A classic method for accomplishing the goal of self-synchronization of binary N-tuples is to initiate each binary code block of length N with a synchronization sequence (or sync sequence, for short), a binary sequence of length, say, m that identifies the beginning of a code block. Each binary N-tuple codeword will consist of a common sync sequence of length m followed by a binary K-tuple with $K = N - m$. To ensure that the sync sequence always identifies the first m symbols of a binary code block, we no longer allow the remaining K binary symbols to be unconstrained — we now prohibit the appearance of the sync sequence

anywhere within the binary K-tuple that follows a sync sequence. Furthermore, for any $\ell = 1, 2, \ldots, m-1$ we cannot allow the final ℓ symbols of a sync m-tuple followed by the first $m-\ell$ binary symbols of any allowed binary K-tuple (that is, any binary K-tuple appearing in the last K positions of an allowed binary N-tuple index) to equal the sync sequence. This last problem is easy to avoid. If m is even, then choose the first $m/2$ symbols of the sync sequence to be 0s and the remainder 1s. If m is odd, then choose the first $(m-1)/2$ symbols to be 0s and the remainder 1s. If ℓ is greater than $m/2$ for m even or $(m-1)/2$ if m is odd, then it is not possible for an overlap of sync and codeword to be mistaken for a sync since the possible false alarm begins with a 1 while a real sync must begin with a 0. if ℓ falls in the first half of a true sync, there will be insufficient 0s in the k-tuple to be mistaken for a sync. Thus we need only be concerned about avoiding a sync sequence inside a binary K-tuple following a sync.

A sync sequence can occur in any of $K - m = N - 2m$ positions in a binary K-tuple, and all of the 2^{N-2m} binary K tuples containing a sync in any of the $N - 2m$ possible positions are disallowed from the index set. After removing all of these disallowed sequences there will be at least $2^K - (N-2m)2^{N-2m}$ K-tuples remaining for indexing the codebook G_n. Note that we have overcounted the number of sequences removed so that we have at least M remaining since K-tuples with two or more sync sequences within them get removed multiple times. Thus the condition required for ensuring that there are enough indices for the words in G_n is

$$2^K - (N-2m)2^{N-2m} \geq 2^{n(1+\epsilon)} \tag{6.13}$$

To relate ϵ, n, k chosen previously to parse the encoder block, we now derive the necessary conditions for K and m for obtaining arbitrarily small average probability of error. Analogous to the synchronous case, fix $\delta_1 > 0$ to be chosen shortly and define the sync sequence length similarly to (6.8) by

$$m = m(N) = \lceil \delta_1 N \rceil \tag{6.14}$$

and set $K = N - m$. As earlier,

$$\begin{aligned} \delta_1 N \leq m \leq \delta_1 N + 1 \\ (1-\delta_1)N - 1 \leq K \leq N(1-\delta_1). \end{aligned}$$

From (6.13) and the definitions we have the inequalities

$$\begin{aligned} 2^K - (N-2m)2^{N-2m} &\geq 2^{N(1-\delta_1)} - (N - 2\delta_1 N - 2)2^{N-2\delta_1 N-2} \\ 2^{(1-\epsilon)N(1+\epsilon)} &\geq 2^{n(1+\epsilon)} \end{aligned}$$

and hence (6.13) will be satisfied and there will be sufficient indices for all of the vectors in G_n if

$$2^{N(1-\delta_1)} - (N - 2\delta_1 N - 2)2^{N-2\delta_1 N-2} \geq 2^{(1-\epsilon^2)N}$$

or

$$2^{-N\delta_1} \geq 2^{-\epsilon^2 N} + (N - 2\delta_1 N - 2)2^{-2\delta_1 N-2}$$

or

$$1 \geq 2^{(\delta_1-\epsilon^2)N} + [N(1 - 2\delta_1 N) - 2]2^{-\delta_1 N-2}.$$

If we choose $\delta_1 < \sqrt{\epsilon}$, then the term on the right goes to zero with N and hence for sufficiently large blocklength N there are sufficient binary N tuples in a self-synchronized code for all vectors in G_n. The analysis of the mean probability of error follows exactly as before. Note that δ_0 yields the fraction of symbols k in the initial and ignored symbols in the input source word, while δ_1 yielded the fraction of initial symbols constituting the sync sequence in the output or encoded binary word.

Sliding-Block Code

Let N and $\epsilon > 0$ remain as before, where now N is chosen large enough to ensure that $\mu_{X^n}(G_n) \leq \epsilon$ in the previous subsections and construct an asynchronous block code of blocklength N as there described. The symbols k and m retain their meaning as the length of the input and output prefixes. Fix $\delta_2 > 0$, and use Lemmas 2.11–2.12 to construct a Rohlin tower with base F, height N, and $\mu_X(G) \leq \delta_2$ having the properties of the lemmas. In particular assume that the finite partition considered is

$$\mathcal{P} = \bigvee_{i=0}^{N-1} T^{-i}\mathcal{P}_0,$$

where $\mathcal{P}_0$ is the zero-time partition for the finite-alphabet source, that is, if the stationary random process $\{X_n\}$ has alphabet $A = \{a_i; i = 0, 1 \ldots, \|A\| - 1\}$ $\mathcal{P}_0 = \{\{x : X_0(x) = x_0 = a_i\}; i = 0, 1 \ldots, \|A\| - 1\}$. Thus the atoms of $\mathcal{P}$ correspond to all sequences having initial N coordinates $x^N = a^N$, for some $a^N \in A^N$. In other words, the atoms are N-dimensional thin cylinders.

The sliding-block encoder operates as follows to map a sequence x into a binary symbol. If $x \in F$, then use the asynchronous block code to map x^N into a binary N-tuple n^N and put out the first symbol b_0. This will be the first symbol in the sync sequence. If $x \in TF$, then $T^{-1}x \in F$. Apply the block code to $x_{-1}^N = (x_{-1}, x_0, \ldots, x_{N-2})$ to obtain b^N and put out the second symbol b_1. Continue in this way: if $x \in T^iF$ for $i = 0, 1, \ldots, N-1$, then $T^{-i}x \in F$ and apply the block code to x_{-i}^N to produce b^N and put out the ith symbol b_i. Lastly, if $x \in G$, put out a 0 (that is, an arbitrary symbol). This defines a stationary encoder for all infinite input

sequences x. Since F is measurable with respect to a finite window and block codes are used, the stationary code is described by a finite-length sliding-block code.

The sliding-block decoder operates essentially in the same manner as the asynchronous block code decoder. Suppose that the sync sequence is a binary m-tuple s^m. Given a received sequence b, look for an $i = 0, 1, \ldots, N-1$ for which $b_{-i}^m = s^m$ (there can be at most one). If there is no such i, put out an arbitrary fixed reference symbol, say b^*. If there is a match with a sync, form the binary N-tuple b_{-i}^N and use the block decoder to map this into a reproduction vector $\hat{x}^N$. Put out $\hat{x}_i$. Recall that $\hat{x}^N$ will be a concatenation of k 0s followed by an n-tuple in G_n, the collection of entropy-typical source vectors.

In a nutshell, the encoder uses the base of the Rohlin tower to initiate a block coding, and usually the block code will be used repetitively until eventually some spacing is thrown in to make things stationary. The resulting binary codewords all have a unique prefix that can occur only at the beginning of a code block.

Consider the error probability resulting from this sliding-block code. Let $\hat{X}_n$ denote the process resulting from encoding and decoding as above and let $\{X_0 \neq \hat{X}_0\}$ be short hand for the set of sequences $\{x : X_0(x) \neq \hat{X}_0(x)\}$, where $X_0(x) = x_0$ is just the coordinate function, and $\hat{X}_0(x)$ is the output at time 0 of the cascade of the sliding-block encoder and decoder. Then using total probability

$$\begin{aligned} P_e &= \Pr(X_0 \neq \hat{X}_0) = \mu_X(\{X_0 \neq \hat{X}_0\}) \\ &= \mu_X(\{X_0 \neq \hat{X}_0\} \cap G) + \sum_{i=0}^{N-1} \mu_X(\{X_0 \neq \hat{X}_0\} \cap T^i F). \end{aligned}$$

Since

$$\mu_X(\{X_0 \neq \hat{X}_0\} \cap G) \leq \mu_X(G_n) \leq \delta_2$$

and

$$\begin{aligned} \mu_X(\{X_0 \neq \hat{X}_0\} \cap T^i F) &= \mu_X(\{x : X_0(x) \neq \hat{X}_0(x)\} \cap T^i F) \\ &= \mu_X(T^{-i}\{x : X_0(x) \neq \hat{X}_0(x)\} \cap F) \\ &= \mu_X(\{x : X_0(T^i x) \neq \hat{X}_0(T^i x)\} \cap F) \\ &= \mu_X(\{X_i \neq \hat{X}_i\} \cap F) \end{aligned}$$

using the stationarity of the process distribution and codes, we have that

$$
\begin{aligned}
P_e &\le \delta_2 + \sum_{i=0}^{N-1} \mu_X(\{X_i \ne \hat{X}_i\} \cap F) \\
&\le \delta_2 + \sum_{i=0}^{k-1} \mu_X(\{X_i \ne \hat{X}_i\} \cap F) + \sum_{i=k}^{N-1} \mu_X(\{X_i \ne \hat{X}_i\} \cap F) \\
&\le \delta_2 + \sum_{i=0}^{k-1} \mu_X(F) + \sum_{i=k}^{N-1} \mu_X(\{X_k^n \ne \hat{X}_k^n\} \cap F) \\
&\le \delta_2 + \frac{k}{N} + \sum_{i=k}^{N-1} \mu_X(\{X_k^n \notin G_n\} \cap F) \\
&\le \delta_2 + \epsilon + \frac{1}{N} + \sum_{i=k}^{N-1} \mu_X(\{X_k^n \notin G_n\} \cap F).
\end{aligned}
$$

Lemma 2.12 implies that for any x^N,

$$\mu_X(\{X^N = x^n\} \cap F) \le \frac{1}{N}\mu_X(\{X^N = x^n\})$$

and hence

$$
\begin{aligned}
\mu_X(\{X_k^n \notin G_n\} \cap F) &= \sum_{x^N : x_k^n \notin G_n} \mu_X(\{X^N = x^n\} \cap F) \\
&\le \sum_{x^N : x_k^n \notin G_n} \frac{1}{N}\mu_X(\{X^N = x^n\}) \\
&= \frac{1}{N}\mu_X(\{X_k^n \notin G_n\}) = \frac{1}{N}\mu_X(\{X^n \notin G_n\}) \le \frac{\epsilon}{N}.
\end{aligned}
$$

Thus

$$P_e \le \delta_2 + \epsilon + \frac{1}{N} + \frac{N-k}{N}\epsilon \le \delta_2 + 2\epsilon,$$

which can be made as small as desired by suitable choices of ϵ, δ_2.

Summarizing, given any finite-alphabet stationary and ergodic source X with entropy rate 1 bit per sample and $\epsilon > 0$, a sliding-block encoder with binary outputs and a sliding-block decoder mapping binary sequences into the source alphabet (both of finite length) can be constructed so that $P_e \le \epsilon$. This implies that *asymptotically optimal* encoders f_n and decoder g_n can be constructed with resulting probability of error $P_e(f_n, g_n)$ going to 0 as $n \to \infty$.

This solves the theoretical problem of showing that a stationary and ergodic process with entropy rate 1 can be coded into bits and decoded in a stationary way so that the original source can be reconstructed without error. This can be viewed as a stationary coding analog of Shannon's noiseless or lossless source coding theorem for variable-length block

codes (which are not stationary). The stationary coding leads to some additional properties, which are explored next.

6.6 Asymptotically Optimal Almost Lossless Codes

Continuing with the almost lossless codes of the previous section, suppose that X is a stationary and ergodic finite-alphabet source with entropy rate 1 bit per symbol. Suppose that we construct a sequence of finite-length sliding block encoders f_n and decoders g_n which result in encoded processes $U^{(n)}$ and decoded reproduction processes $\hat{X}^{(n)}$. Since coding can only reduce entropy rate, we have immediately that

$$\overline{H}(X) = 1 \geq \overline{H}(U^{(n)}) \geq \overline{H}(\hat{X}^{(n)}). \tag{6.15}$$

Since the codes are asymptotically optimal,

$$\lim_{n\to\infty} \Pr(X_0 \neq \hat{X}_0^{(n)}) = 0. \tag{6.16}$$

The codes form a coupling between the input and output, and the probability of error is simply the expected Hamming distortion between the pair. This distortion is bound below by Ornstein's d-bar distance between the processes, which means that

$$\lim_{n\to\infty} \overline{d}(X, \hat{X}^{(n)}) = 0. \tag{6.17}$$

Thus the output process converges in d-bar to the original source.

Eq. (6.16) implies from the process version of the Fano inequality that

$$\lim_{n\to\infty} \overline{H}(\hat{X}^{(n)}) = \overline{H}(X) = 1. \tag{6.18}$$

From (6.15), this forces

$$\lim_{n\to\infty} \overline{H}(U^{(n)}) = 1. \tag{6.19}$$

This suggests that the binary encoded process is looking more and more like coin flips as n grows. Marton's inequality as in Corollary 6.6 provides a rigorous proof of this fact. Suppose that Z is the fair coin flip process, an IID binary equiprobable process. Then from Corollary 6.6

$$\overline{d}(U^{(n)}, Z) \leq \sqrt{\frac{1}{2}(\overline{H}(Z) - \overline{H}(U^{(n)}))}$$

which goes to 0 as $n \to \infty$. Thus the encoded process converges to fair coin flips in d-bar if the codes are asymptotically optimal, and the output of the decoder converges to the original source in d-bar.

Together these facts provide an approximate variation on the Ornstein isomorphism and Sinai theorems of ergodic theory. The isomorphism theorem states that a B-process with entropy rate 1 can be coded into coin flips in an invertible manner. We have just seen that we can map the original process into a process that is very close to coin flips in d-bar, and then we can "invert" the mapping by another mapping which produces a process very close to the original in d-bar. Thus almost lossless source coding can be interpreted as an approximate version of the isomorphism theorem, and the isomorphism theorem can be interpreted as a limiting version of the lossless coding result. The source coding result holds generally for stationary and ergodic processes, but turning it into the vastly stronger isomorphism theorem requires B-processes. The Sinai theorem states that we can model a process with prescribed distributions and entropy rate 1 by a stationary coding of coin flips. We have shown that a stationary coding of something d-bar close to coin flips can be used to obtain something d-bar close to a prescribed process. This can be interpreted as a simulation result, generating a desired process from coin flips.

6.7 Modeling and Simulation

Random processes and dynamical systems can be used to model real world phenomena in the sense that one can use statistical methods to fit a probabilistic description to observed measurements. Such models are based on an assumption that observed relative frequencies of measurements predict future behavior, the fundamental idea of the ergodic theorem and the notion of AMS processes. Often models can be of a specific form, such as Gaussian or Poisson, based on the observed or assumed physics describing the production of the measured quantities. It also often occurs that the form of a model is assumed simply for convenience, and its suitability for the signals in question may be controversial. In some situations one might wish to place constraints on the model, but not assume particular distributions. For example one might wish to allow only models of a particularly simple form with useful properties such as *B*-processes, processes formed by stationary coding or filtering of IID processes, or autoregressive processes, processes formed by linear filtering an IID process using a poles-only or autoregressive filter. We shall later see (in Lemma 14.13) that a source can be communicated with arbitrarily small distortion through a noisy channel if the source has entropy rate less than a quantity determined by the channel (called the channel capacity) and if the source is totally ergodic. This fundamental result of information theory implies that a particularly useful class of models is the class with an entropy constraint. Furthermore, totally

ergodic sources such as B-processes are amenable to reliable communication and hence a natural choice of model.

This raises the issue of how good a model is as a fit to a "real" process, and process distortion measures provide a means of quantifying just how good a model approximates a target process. This situation can be idealized by a process distortion between a "true" distribution and the model distribution, but this assumes the existence of the former. Theorem 5.2 provides a practical means of estimating a process distortion by finding the best match between a sequence produced by the target process (which will produce a frequency-typical sequence with probability 1) and the model's collection of frequency-typical sequences. A minimum distortion selection between a fixed sequence and a member of a collection of sequences resembles the encoder we will encounter in source coding in Chapter 12, so that the considerations of this section will shed insight on the source coding problem.

With this introduction we formalize two optimization problems describing the fitting of a model from an interesting class of processes to a target process so as to minimize a process distortion between the target and the class. It is assumed that the target process distribution exists, but it should be kept in mind that in the case of stationary and ergodic processes, the process distortion can be estimated by finding the best match between an example target sequence and the collection of frequency-typical model sequences.

Suppose that μ_X is the distribution of a stationary source $\{X_n\}$. For some class of random processes $\mathcal{P}$ define

$$\overline{\rho}(\mu_X, \mathcal{P}) = \inf_{\mu_Y \in \mathcal{P}} \overline{\rho}(\mu_X, \mu_Y). \tag{6.20}$$

The previous discussion suggests the classes

$$\mathcal{P}(R) = \{\mu_Y : \overline{H}(Y) \le R\} \tag{6.21}$$

$$\mathcal{P}_B(R) = \{B\text{-processes } \mu_Y : \overline{H}(Y) \le R\} \tag{6.22}$$

where $R \ge 0$.

The first optimization problem originated in [65] and will be seen in Chapter 12 to provide a geometric interpretation of Shannon source coding. The second originated in [53], where it was dubbed the "simulation problem" because of its goal of generating a good model of a target process as a sliding-block coding of a simple finite alphabet IID process such as fair coin flips. The sliding-block simulating code was shown to provide a good decoder in a source coding system. Both of these results will be considered later. They are introduced here as a natural combination of distortion and entropy considerations. Since $\mathcal{P}_B(R) \subset \mathcal{P}(R)$, $\overline{\rho}(\mu_X, \mathcal{P}_B(R)) \ge \overline{\rho}(\mu_X, \mathcal{P}(R))$.

Of more practical interest than simply modeling or simulating a random process as a B-process is the possibility of using a stationary encoding of a particular IID process such as fair coin flips or dice rolls. For example, suppose that one is given an IID process Z with distribution μ_Z and entropy rate $\overline{H}(Z)$ and one wishes to simulate a random process X with distribution μ_X by applying a stationary code f to Z to produce a process $\tilde{X} = f(Z)$ with entropy rate $\overline{H}(\tilde{X}) \le \overline{H}(Z)$ which is as close as possible to X in rho-bar:

$$\Delta_{X|Z} = \inf_f \overline{\rho}(\mu_X, \mu_{f(Z)}). \tag{6.23}$$

From the definitions,

$$\Delta_{X|Z} \ge \overline{\rho}(\mu_X, \mathcal{P}_B(\overline{H}(Z))) \tag{6.24}$$

since the optimization over B-processes formed by stationary codings of Z is a more constrained optimization. Suppose, however, that X is itself a B-process and that $\tilde{X}$ is a B-process approximately solving the minimization of $\overline{\rho}(\mu_X, \mathcal{P}_B(\overline{H}(Z)))$ so that $\tilde{X}$ is a B-process with $\overline{H}(\tilde{X}) \le \overline{H}(Z)$ and $\overline{\rho}(\mu_X, \mu_{\tilde{X}}) \le \epsilon$ for some small $\epsilon > 0$. Then Sinai's theorem implies that $\tilde{X}$ (or rather a process equivalent to $\tilde{X}$) can be obtained as a stationary coding of Z and hence

$$\Delta_{X|Z} = \overline{\rho}(\mu_X, \mathcal{P}_B(\overline{H}(Z))).$$

If $R = \overline{H}(Z)$, this means that

$$\Delta_{X|Z} = \overline{\rho}(\mu_X, \mathcal{P}_B(R)) \equiv \Delta_X(R), \tag{6.25}$$

a quantity that depends only on R and not on the structure of Z other than its entropy rate!

Chapter 7
Relative Entropy

Abstract A variety of information measures have been introduced for finite alphabet random variables, vectors, and processes: entropy, mutual information, relative entropy, conditional entropy, and conditional mutual information. All of these can be expressed in terms of divergence and hence the generalization of these definitions and their properties to infinite alphabets will follow from a general definition of divergence. In this chapter the definition and properties of divergence in this general setting are developed, including the formulas for evaluating divergence as an expectation of information density and as a limit of divergences of finite codings. We also develop several inequalities for and asymptotic properties of divergence. These results provide the groundwork needed for generalizing the ergodic theorems of information theory from finite to standard alphabets. The general definitions of entropy and information measures originated in the pioneering work of Kolmogorov and his colleagues Gelfand, Yaglom, Dobrushin, and Pinsker

7.1 Divergence

Given a probability space $(\Omega, \mathcal{B}, P)$ (not necessarily with finite alphabet) and another probability measure M on the same space, define the *relative entropy* or *divergence of P with respect to M* by

$$D(P\|M) = \sup_{\mathcal{Q}} H_{P\|M}(\mathcal{Q}) = \sup_{f} D(P_f\|M_f), \tag{7.1}$$

where the first supremum is over all finite measurable partitions $\mathcal{Q}$ of Ω and the second is over all finite alphabet measurements on Ω. The two forms have the same interpretation: the divergence is the supremum of the relative entropies or divergences obtainable by finite alphabet codings of the sample space. The partition form is perhaps more common

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when considering divergence *per se*, but the measurement or code form is usually more intuitive when considering entropy and information. This section is devoted to developing the basic properties of divergence, all of which will yield immediate corollaries for the measures of information.

The first result is a generalization of the divergence inequality that is a trivial consequence of the definition and the finite alphabet special case.

Lemma 7.1. *The Divergence Inequality:*
For any two probability measures P and M

$$D(P\|M) \geq 0$$

with equality if and only if $P = M$.

Proof. Given any partition $\mathcal{Q}$, Theorem 3.1 implies that

$$\sum_{Q\in\mathcal{Q}} P(Q) \ln \frac{P(Q)}{M(Q)} \geq 0$$

with equality if and only if $P(Q) = M(Q)$ for all atoms Q of the partition. Since $D(P\|Q)$ is the supremum over all such partitions, it is also nonnegative. It can be 0 only if P and M assign the same probabilities to all atoms in all partitions (the supremum is 0 only if the above sum is 0 for all partitions) and hence the divergence is 0 only if the measures are identical. □

As in the finite alphabet case, Lemma 7.1 justifies interpreting divergence as a form of distance or dissimilarity between two probability measures. It is not a true distance or metric in the mathematical sense since it is not symmetric and it does not satisfy the triangle inequality. Since it is nonnegative and equals zero only if two measures are identical, the divergence is a *distortion measure* on probability distributions as considered in Chapter 5. This view often provides interpretations of the basic properties of divergence. We shall develop several relations between the divergence and other distance measures. The reader is referred to Csiszár [26] for a development of the distance-like properties of divergence.

As the supremum definition of divergence in the general case permits an easy generalization of the divergence inequality, it also permits an easy generalization of the basic convexity property of Corollary 3.5.

Lemma 7.2. *The divergence $D(P\|M)$ is a convex function of the pair of probability measures (P, M).*

The lemma can also be proved using the integral representation of divergence as in Csiszár [25].

The following two lemmas provide means for computing divergences and studying their behavior. The first result shows that the supremum can be confined to partitions with atoms in a generating field. This will provide a means for computing divergences by approximation or limits. The result is due to Dobrushin and is referred to as Dobrushin's theorem. The second result shows that the divergence can be evaluated as the expectation of an entropy density defined as the logarithm of the Radon-Nikodym derivative of one measure relative to the other. This result is due to Gelfand, Yaglom, and Perez. The proofs largely follow the translator's remarks in Chapter 2 of Pinsker [150] (which in turn follows Dobrushin [32]).

Lemma 7.3. *Suppose that* $(\Omega, \mathcal{B})$ *is a measurable space where* $\mathcal{B}$ *is generated by a field* $\mathcal{F}$, $\mathcal{B} = \sigma(\mathcal{F})$. *Then if* P *and* M *are two probability measures on this space,*

$$D(P\|M) = \sup_{\mathcal{Q}\subset\mathcal{F}} H_{P\|M}(\mathcal{Q}).$$

Proof. From the definition of divergence, the right-hand term above is clearly less than or equal to the divergence. If P is not absolutely continuous with respect to M, then we can find a set F such that $M(F) = 0$ but $P(F) \neq 0$ and hence the divergence is infinite. Approximating this event by a field element F_0 by applying Theorem 1.1 simultaneously to M and G will yield a partition $\{F_0, F_0^c\}$ for which the right hand side of the previous equation is arbitrarily large. Hence the lemma holds for this case. Henceforth assume that $M \gg P$.

Fix $\epsilon > 0$ and suppose that a partition $\mathcal{Q} = \{Q_1, \cdots, Q_K\}$ yields a relative entropy close to the divergence, that is,

$$H_{P\|M}(\mathcal{Q}) = \sum_{i=1}^{K} P(Q_i) \ln \frac{P(Q_i)}{M(Q_i)} \geq D(P\|M) - \epsilon/2.$$

We will show that there is a partition, say $\mathcal{Q}'$ with atoms in $\mathcal{F}$ which has almost the same relative entropy, which will prove the lemma. First observe that $P(Q)\ln[P(Q)/M(Q)]$ is a continuous function of $P(Q)$ and $M(Q)$ in the sense that given $\epsilon/(2K)$ there is a sufficiently small $\delta > 0$ such that if $|P(Q) - P(Q')| \leq \delta$ and $|M(Q) - M(Q')| \leq \delta$, then provided $M(Q) \neq 0$

$$|P(Q) \ln \frac{P(Q)}{M(Q)} - P(Q') \ln \frac{P(Q')}{M(Q')}| \leq \frac{\epsilon}{2K}.$$

If we can find a partition $\mathcal{Q}'$ with atoms in $\mathcal{F}$ such that

$$|P(Q_i') - P(Q_i)| \leq \delta,\ |M(Q_i') - M(Q_i)| \leq \delta,\ i = 1, \cdots, K, \tag{7.2}$$

then

$$|H_{P\|M}(\mathcal{Q}') - H_{P\|M}(\mathcal{Q})| \leq \sum_i |P(Q_i)\ln\frac{P(Q_i)}{M(Q_i)} - P(Q_i')\ln\frac{P(Q_i')}{M(Q_i')}|$$
$$\leq K\frac{\epsilon}{2K} = \frac{\epsilon}{2}$$

and hence

$$H_{P\|M}(\mathcal{Q}') \geq D(P\|M) - \epsilon$$

which will prove the lemma. To find the partition $\mathcal{Q}'$ satisfying (7.2), let m be the mixture measure $P/2 + M/2$. As in the proof of Lemma 5.1, we can find a partition $\mathcal{Q}' \subset \mathcal{F}$ such that $m(Q_i \Delta Q_i') \leq K^2\gamma$ for $i = 1, 2, \cdots, K$, which implies that

$$P(Q_i\Delta Q_i') \leq 2K^2\gamma \text{ and } M(Q_i\Delta Q_i') \leq 2K^2\gamma;\ i = 1, 2, \cdots, K.$$

If we now choose γ so small that $2K^2\gamma \leq \delta$, then (7.2) and hence the lemma follow from the above and the fact that

$$|P(F) - P(G)| \leq P(F\Delta G). \tag{7.3}$$

□

Lemma 7.4. *Given two probability measures P and M on a common measurable space $(\Omega, \mathcal{B})$, if P is not absolutely continuous with respect to M, then*

$$D(P\|M) = \infty.$$

If $P \ll M$ (e.g., if $D(P\|M) < \infty$), then the Radon-Nikodym derivative $f = dP/dM$ exists and

$$D(P\|M) = \int \ln f(\omega)dP(\omega) = \int f(\omega)\ln f(\omega)dM(\omega).$$

The quantity $\ln f$ (if it exists) is called the *entropy density* or *relative entropy density* of P with respect to M.

Proof. The first statement was shown in the proof of the previous lemma. If P is not absolutely continuous with respect to M, then there is a set Q such that $M(Q) = 0$ and $P(Q) > 0$. The relative entropy for the partition $\mathcal{Q} = \{Q, Q^c\}$ is then infinite, and hence so is the divergence.

Assume that $P \ll M$ and let $f = dP/dM$. Suppose that Q is an event for which $M(Q) > 0$ and consider the conditional cumulative distribution function for the real random variable f given that $\omega \in Q$:

$$F_Q(u) = \frac{M(\{f < u\} \bigcap Q)}{M(Q)}; u \in (-\infty, \infty).$$

Observe that the expectation with respect to this distribution is

$$E_M(f|Q) = \int_0^\infty u\, dF_Q(u) = \frac{1}{M(Q)} \int_Q f(\omega)\, dM(\omega) = \frac{P(Q)}{M(Q)}.$$

We also have that

$$\int_0^\infty u \ln u\, dF_Q(u) = \frac{1}{M(Q)} \int_Q f(\omega) \ln f(\omega)\, dM(\omega),$$

where the existence of the integral is ensured by the fact that $u \ln u \geq -e^{-1}$.

Applying Jensen's inequality to the convex $\bigcup$ function $u \ln u$ yields the inequality

$$\begin{aligned}
\frac{1}{M(Q)} \int_Q \ln f(\omega)\, dP(\omega) &= \frac{1}{M(Q)} \int_Q f(\omega) \ln f(\omega)\, dM(\omega) \\
&= \int_0^\infty u \ln u\, dF_Q(u) \\
&\geq [\int_0^\infty u\, dF_Q(u)] \ln[\int_0^\infty u\, dF_Q(u)] \\
&= \frac{P(Q)}{M(Q)} \ln \frac{P(Q)}{M(Q)}.
\end{aligned}$$

We therefore have that for any event Q with $M(Q) > 0$ that

$$\int_Q \ln f(\omega)\, dP(\omega) \geq P(Q) \ln \frac{P(Q)}{M(Q)}. \tag{7.4}$$

Let $\mathcal{Q} = \{Q_i\}$ be a finite partition and we have

$$\begin{aligned}
\int \ln f(\omega) dP(\omega) &= \sum_i \int_{Q_i} \ln f(\omega)\, dP(\omega) \\
&\geq \sum_{i:P(Q_i)\neq 0} \int_{Q_i} \ln f(\omega)\, dP(\omega) \\
&= \sum_i P(Q_i) \ln \frac{P(Q_i)}{M(Q_i)},
\end{aligned}$$

where the inequality follows from (7.4) since $P(Q_i) \neq 0$ implies that $M(Q_i) \neq 0$ since $M \gg P$. This proves that

$$D(P\|M) \leq \int \ln f(\omega)\, dP(\omega).$$

To obtain the converse inequality, let q_n denote the asymptotically accurate quantizers of Section 1.5. From (1.23)

$$\int \ln f(\omega)\, dP(\omega) = \lim_{n\to\infty} \int q_n(\ln f(\omega))\, dP(\omega).$$

For fixed n the quantizer q_n induces a partition of Ω into $2n2^n + 1$ atoms $\mathcal{Q}$. In particular, there are $2n2^n - 1$ "good" atoms such that for ω, ω' inside the atoms we have that $|\ln f(\omega) - \ln f(\omega')| \leq 2^{-(n-1)}$. The remaining two atoms group ω for which $\ln f(\omega) \geq n$ or $\ln f(\omega) < -n$. Defining the shorthand $P(\ln f < -n) = P(\{\omega : \ln f(\omega) < -n\})$, we have then that

$$\sum_{Q \in \mathcal{Q}} P(Q) \ln \frac{P(Q)}{M(Q)} = \sum_{\text{good } Q} P(Q) \ln \frac{P(Q)}{M(Q)} + P(\ln f \geq n) \ln \frac{P(\ln f \geq n)}{M(\ln f \geq n)} + P(\ln f < -n) \ln \frac{P(\ln f < -n)}{M(\ln f < -n)}.$$

The rightmost two terms above are bounded below as

$$\begin{aligned} &P(\ln f \geq n) \ln \frac{P(\ln f \geq n)}{M(\ln f \geq n)} + P(\ln f < -n) \ln \frac{P(\ln f < -n)}{M(\ln f < -n)} \\ &\quad \geq P(\ln f \geq n) \ln P(\ln f \geq n) + P(\ln f < -n) \ln P(\ln f < -n). \end{aligned}$$

Since $P(\ln f \geq n)$ and $P(\ln f < -n) \to 0$ as $n \to \infty$ and since $x \ln x \to 0$ as $x \to 0$, given ϵ we can choose n large enough to ensure that the above term is greater than $-\epsilon$. This yields the lower bound

$$\sum_{Q \in \mathcal{Q}} P(Q) \ln \frac{P(Q)}{M(Q)} \geq \sum_{\text{good } Q} P(Q) \ln \frac{P(Q)}{M(Q)} - \epsilon.$$

Fix a good atom Q and define $\bar{h} = \sup_{\omega \in Q} \ln f(\omega)$ and $\underline{h} = \inf_{\omega \in Q} \ln f(\omega)$ and note that by definition of the good atoms

$$\bar{h} - \underline{h} \leq 2^{-(n-1)}.$$

We now have that

$$P(Q)\bar{h} \geq \int_Q \ln f(\omega)\, dP(\omega)$$

and

$$M(Q)e^{\underline{h}} \leq \int_Q f(\omega) dM(\omega) = P(Q).$$

Combining these yields

$$\begin{aligned} P(Q) \ln \frac{P(Q)}{M(Q)} &\geq P(Q) \ln \frac{P(Q)}{P(Q)e^{-\underline{h}}} = P(Q)\underline{h} \\ &\geq P(Q)(\bar{h} - 2^{-(n-1)}) \\ &\geq \int_Q \ln f(\omega) dP(\omega) - P(Q)2^{-(n-1)}. \end{aligned}$$

Therefore

$$\begin{aligned}\sum_{Q\in\mathcal{Q}} P(Q)\ln\frac{P(Q)}{M(Q)} &\geq \sum_{\text{good } Q} P(Q)\ln\frac{P(Q)}{M(Q)} - \epsilon \\ &\geq \sum_{\text{good } Q}\int_Q \ln f(\omega)\,dP - 2^{-(n-1)} - \epsilon \\ &= \int_{\omega:|\ln f(\omega)|\leq n} \ln f(\omega)\,dP(\omega) - 2^{-(n-1)} - \epsilon.\end{aligned}$$

Since this is true for arbitrarily large n and arbitrarily small ϵ,

$$D(P\|Q) \geq \int \ln f(\omega) dP(\omega),$$

completing the proof of the lemma. □

It is worthwhile to point out two examples for the previous lemma. If P and M are discrete measures with corresponding PMF's p and q, then the Radon-Nikodym derivative is simply $dP/dM(\omega) = p(\omega)/m(\omega)$ and the lemma gives the known formula for the discrete case. If P and M are both probability measures on Euclidean space $\mathcal{R}^n$ and if both measures are absolutely continuous with respect to Lebesgue measure, then there exists a density f called a *probability density function* or *pdf* such that

$$P(F) = \int_F f(x)dx,$$

where dx means $dm(x)$ with m Lebesgue measure. (Lebesgue measure assigns each set its volume.) Similarly, there is a pdf g for M. In this case,

$$D(P\|M) = \int_{\mathcal{R}^n} f(x)\ln\frac{f(x)}{g(x)}dx. \tag{7.5}$$

The following immediate corollary to the previous lemma provides a formula that is occasionally useful for computing divergences.

Corollary 7.1. *Given three probability distributions $M \gg Q \gg P$, then*

$$D(P\|M) = D(P\|Q) + E_P(\ln\frac{dQ}{dM}).$$

Proof. From the chain rule for Radon-Nikodym derivatives (e.g., Lemma 5.7.3 of [55] or Lemma 6.6 of [58])

$$\frac{dP}{dM} = \frac{dP}{dQ}\frac{dQ}{dM}$$

and taking expectations using the previous lemma yields the corollary. □

The next result is a technical result that shows that given a mapping on a space, the divergence between the induced distributions can be computed from the restrictions of the original measures to the sub-σ-field induced by the mapping. As part of the result, the relation between the induced Radon-Nikodym derivative and the original derivative is made explicit.

Recall that if P is a probability measure on a measurable space $(\Omega, \mathcal{B})$ and if $\mathcal{F}$ is a sub-σ-field of $\mathcal{B}$, then the restriction $P_{\mathcal{F}}$ of P to $\mathcal{F}$ is the probability measure on the measurable space $(\Omega, \mathcal{F})$ defined by $P_{\mathcal{F}}(G) = P(G)$, for all $G \in \mathcal{F}$. In other words, we can use either the probability measures on the new space or the restrictions of the probability measures on the old space to compute the divergence. This motivates considering the properties of divergences of restrictions of measures, a useful generality in that it simplifies proofs. The following lemma can be viewed as a bookkeeping result relating the divergence and the Radon-Nikodym derivatives in the two spaces.

Lemma 7.5. *(a) Suppose that M, P are two probability measures on a space $(\Omega, \mathcal{B})$ and that X is a measurement mapping this space into $(A, \mathcal{A})$. Let P_X and M_X denote the induced distributions (measures on $(A, \mathcal{A})$) and let $P_{\sigma(X)}$ and $M_{\sigma(X)}$ denote the restrictions of P and M to $\sigma(X)$, the sub-σ-field of $\mathcal{B}$ generated by X. Then*

$$D(P_X \| M_X) = D(P_{\sigma(X)} \| M_{\sigma(X)}).$$

If the Radon-Nikodym derivative $f = dP_X/dM_X$ exists (e.g., the above divergence is finite), then define the function $f(X) : \Omega \to [0, \infty)$ by

$$f(X)(\omega) = f(X(\omega)) = \frac{dP_X}{dM_X}(X(\omega));$$

then with probability 1 under both M and P

$$f(X) = \frac{dP_{\sigma(X)}}{dM_{\sigma(X)}}.$$

(b) Suppose that $P \ll M$. Then for any sub-σ-field $\mathcal{F}$ of $\mathcal{B}$, we have that

$$\frac{dP_{\mathcal{F}}}{dM_{\mathcal{F}}} = E_M\left(\frac{dP}{dM} | \mathcal{F}\right).$$

Thus the Radon-Nikodym derivative for the restrictions is just the conditional expectation of the original Radon-Nikodym derivative.

Proof. The proof is mostly algebra: $D(P_{\sigma(X)} \| M_{\sigma(X)})$ is the supremum over all finite partitions $\mathcal{Q}$ with elements in $\sigma(X)$ of the relative entropy

$H_{P_{\sigma(X)}\|M_{\sigma(X)}}(\mathcal{Q})$. Each element $Q \in \mathcal{Q} \subset \sigma(X)$ corresponds to a unique set $Q' \in \mathcal{A}$ via $Q = X^{-1}(Q')$ and hence to each $\mathcal{Q} \subset \sigma(X)$ there is a corresponding partition $\mathcal{Q}' \subset \mathcal{A}$. The corresponding relative entropies are equal, however, since

$$\begin{aligned} H_{P_X\|M_X}(\mathcal{Q}') &= \sum_{Q' \in \mathcal{Q}'} P_f(Q') \ln \frac{P_X(Q')}{M_X(Q')} \\ &= \sum_{Q' \in \mathcal{Q}'} P(X^{-1}(Q')) \ln \frac{P(X^{-1}(Q'))}{M(X^{-1}(Q'))} \\ &= \sum_{Q \in \mathcal{Q}} P_X(Q) \ln \frac{P_X(Q)}{M_X(Q)} \\ &= H_{P_{\sigma(X)}\|M_{\sigma(X)}}(\mathcal{Q}). \end{aligned}$$

Taking the supremum over the partitions proves that the divergences are equal. If the derivative is $f = dP_X/dM_X$, then $f(X)$ is measurable since it is a measurable function of a measurable function. In addition, it is measurable with respect to $\sigma(X)$ since it depends on ω only through $X(\omega)$. For any $F \in \sigma(X)$ there is a $G \in \mathcal{A}$ such that $F = X^{-1}(G)$ and

$$\int_F f(X) dM_{\sigma(X)} = \int_F f(X) dM = \int_G f dM_X$$

from the change of variables formula (see, e.g., Lemma 4.4.7 of [55] or Lemma 5.12 of [58]). Thus

$$\int_F f(X) dM_{\sigma(X)} = P_X(G) = P_{\sigma(X)}(X^{-1}(G)) = P_{\sigma(X)}(F),$$

which proves that $f(X)$ is indeed the claimed derivative with probability 1 under M and hence also under P.

The variation quoted in part (b) is proved by direct verification using iterated expectation. If $G \in \mathcal{F}$, then using iterated expectation we have that

$$\int_G E_M(\frac{dP}{dM}|\mathcal{F})\, dM_{\mathcal{F}} = \int E_M(1_G \frac{dP}{dM}|\mathcal{F})\, dM_{\mathcal{F}}.$$

Since the argument of the integrand is $\mathcal{F}$-measurable (see, e.g., Lemma 5.3.1 of [55] or Lemma 6.3 of [58]), invoking iterated expectation (e.g., Corollary 5.9.3 of [55] or Corollary 6.5 of [58]) yields

$$\begin{aligned} \int_G E_M(\frac{dP}{dM}|\mathcal{F})\, dM_{\mathcal{F}} &= \int E_M(1_G \frac{dP}{dM}|\mathcal{F})\, dM \\ &= E(1_G \frac{dP}{dM}) = P(G) = P_{\mathcal{F}}(G), \end{aligned}$$

proving that the conditional expectation is the claimed derivative. □

Part (b) of the lemma was pointed out to the author by Paul Algoet.

Having argued above that restrictions of measures are useful when finding divergences of random variables, we provide a key trick for treating such restrictions.

Lemma 7.6. *Let $M \gg P$ be two measures on a space $(\Omega, \mathcal{B})$. Suppose that $\mathcal{F}$ is a sub-σ-field and that $P_{\mathcal{F}}$ and $M_{\mathcal{F}}$ are the restrictions of P and M to $\mathcal{F}$ Then there is a measure S such that $M \gg S \gg P$ and*

$$\frac{dP}{dS} = \frac{dP/dM}{dP_{\mathcal{F}}/dM_{\mathcal{F}}},$$

$$\frac{dS}{dM} = \frac{dP_{\mathcal{F}}}{dM_{\mathcal{F}}},$$

and

$$D(P\|S) + D(P_{\mathcal{F}}\|M_{\mathcal{F}}) = D(P\|M). \tag{7.6}$$

Proof. If $M \gg P$, then clearly $M_{\mathcal{F}} \gg P_{\mathcal{F}}$ and hence the appropriate Radon-Nikodym derivatives exist. Define the set function S by

$$S(F) = \int_F \frac{dP_{\mathcal{F}}}{dM_{\mathcal{F}}}\, dM = \int_F E_M(\frac{dP}{dM}|\mathcal{F})\, dM,$$

using part (b) of the previous lemma. Thus $M \gg S$ and $dS/dM = dP_{\mathcal{F}}/dM_{\mathcal{F}}$. Observe that for $F \in \mathcal{F}$, iterated expectation implies that

$$\begin{aligned} S(F) = E_M(E_M(1_F \frac{dP}{dM}|\mathcal{F})) &= E_M(1_F \frac{dP}{dM}) \\ &= P(F) = P_{\mathcal{F}}(F);\ F \in \mathcal{F} \end{aligned}$$

and hence in particular that $S(\Omega)$ is 1 so that $dP_{\mathcal{F}}/dM_{\mathcal{F}}$ is integrable and S is indeed a probability measure on $(\Omega, \mathcal{B})$. (In addition, the restriction of S to $\mathcal{F}$ is just $P_{\mathcal{F}}$.) Define

$$g = \frac{dP/dM}{dP_{\mathcal{F}}/dM_{\mathcal{F}}}.$$

This is well defined since with M probability 1, if the denominator is 0, then so is the numerator. Given $F \in \mathcal{B}$ the Radon-Nikodym theorem (e.g., Theorem 5.6.1 of [55] or Theorem 6.2 of [58]) implies that

$$\int_F g dS = \int 1_F g \frac{dS}{dM} dM = \int 1_F \frac{dP/dM}{dP_{\mathcal{F}}/dM_{\mathcal{F}}} dP_{\mathcal{F}}/dM_{\mathcal{F}} dM = P(F),$$

that is, $P \ll S$ and

$$\frac{dP}{dS} = \frac{dP/dM}{dP_{\mathcal{F}}/dM_{\mathcal{F}}},$$

proving the first part of the lemma. The second part follows by direct verification:

$$\begin{aligned} D(P\|M) = \int \ln \frac{dP}{dM}\, dP &= \int \ln \frac{dP_{\mathcal{F}}}{dM_{\mathcal{F}}}\, dP + \int \ln \frac{dP/dM}{dP_{\mathcal{F}}/dM_{\mathcal{F}}}\, dP \\ &= \int \ln \frac{dP_{\mathcal{F}}}{dM_{\mathcal{F}}}\, dP_{\mathcal{F}} + \int \ln \frac{dP}{dS}\, dP \\ &= D(P_{\mathcal{F}}\|M_{\mathcal{F}}) + D(P\|S). \end{aligned}$$

□

The two previous lemmas and the divergence inequality immediately yield the following result for $M \gg P$. If M does not dominate P, then the result is trivial.

Corollary 7.2. *Given two measures M, P on a space $(\Omega, \mathcal{B})$ and a sub-σ-field $\mathcal{F}$ of $\mathcal{B}$, then*

$$D(P\|M) \geq D(P_{\mathcal{F}}\|M_{\mathcal{F}}).$$

If f is a measurement on the given space, then

$$D(P\|M) \geq D(P_f\|M_f).$$

The result is obvious for finite fields $\mathcal{F}$ or finite alphabet measurements f from the definition of divergence. The general result for arbitrary measurable functions could also have been proved by combining the corresponding finite alphabet result of Corollary 3.2 and an approximation technique. As above, however, we will occasionally get results comparing the divergences of measures and their restrictions by combining the trick of Lemma 7.6 with a result for a single divergence.

The following corollary follows immediately from Lemma 7.3 since the union of a sequence of asymptotically generating sub-σ-fields is a generating field.

Corollary 7.3. *Suppose that M, P are probability measures on a measurable space $(\Omega, \mathcal{B})$ and that $\mathcal{F}_n$ is an asymptotically generating sequence of sub-σ-fields and let P_n and M_n denote the restrictions of P and M to $\mathcal{F}_n$ (e.g., $P_n = P_{\mathcal{F}_n}$). Then*

$$D(P_n\|M_n) \uparrow D(P\|M).$$

There are two useful special cases of the above corollary which follow immediately by specifying a particular sequence of increasing sub-σ-fields. The following two corollaries give these results.

Corollary 7.4. *Let M, P be two probability measures on a measurable space $(\Omega, \mathcal{B})$. Suppose that f is an A-valued measurement on the space.*

Assume that $q_n : A \to A_n$ is a sequence of measurable mappings into finite sets A_n with the property that the sequence of fields $\mathcal{F}_n = \mathcal{F}(q_n(f))$ generated by the sets $\{q_n^{-1}(a);\ a \in A_n\}$ asymptotically generate $\sigma(f)$. (For example, if the original space is standard let $\mathcal{F}_n$ be a basis and let q_n map the points in the ith atom of $\mathcal{F}_n$ into i.) Then

$$D(P_f \| M_f) = \lim_{n\to\infty} D(P_{q_n(f)} \| M_{q_n(f)}).$$

The corollary states that the divergence between two distributions of a random variable can be found as a limit of quantized versions of the random variable. Note that the limit could also be written as

$$\lim_{n\to\infty} H_{P_f \| M_f}(q_n).$$

In the next corollary we consider increasing sequences of random variables instead of increasing sequences of quantizers, that is, more random variables (which need not be finite alphabet) instead of ever finer quantizers. The corollary follows immediately from Corollary 7.3 and Lemma 7.5.

Corollary 7.5. *Suppose that M and P are measures on the sequence space corresponding to outcomes of a sequence of random variables $X_0, X_1, \cdots$ with alphabet A. Let $\mathcal{F}_n = \sigma(X_0, \cdots, X_{n-1})$, which asymptotically generates the σ-field $\sigma(X_0, X_1, \cdots)$. Then*

$$\lim_{n\to\infty} D(P_{X^n} \| M_{X^n}) = D(P \| M).$$

We now develop two fundamental inequalities involving entropy densities and divergence. The first inequality is from Pinsker [150]. The second is an improvement of an inequality of Pinsker [150] by Csiszár [24] and Kullback [105]. The second inequality is more useful when the divergence is small. Coupling these inequalities with the trick of Lemma 7.6 provides a simple generalization of an inequality of [54] and will provide easy proofs of L^1 convergence results for entropy and information densities. Recall from Section 5.9 that given two probability measures M, P on a common measurable space $(\Omega, \mathcal{B})$, the *variation distance* between them is defined by

$$\mathrm{var}(P, M) \equiv \sup_{\mathcal{Q}} \sum_{Q\in\mathcal{Q}} |P(Q) - M(Q)|,$$

where the supremum is over all finite measurable partitions. We will proceed by stating first the end goal — the two inequalities involving divergence — as a lemma, and then state a lemma giving the basic required properties of the variational distance. The lemmas will be proved in a different order.

Lemma 7.7. *Let P and M be two measures on a common probability space $(\Omega, \mathcal{B})$ with $P \ll M$. Let $f = dP/dM$ be the Radon-Nikodym derivative and let $h = \ln f$ be the entropy density. Then*

$$D(P\|M) \leq \int |h| dP \leq D(P\|M) + \frac{2}{e}, \tag{7.7}$$

$$\int |h| dP \leq D(P\|M) + \sqrt{2D(P\|M)}. \tag{7.8}$$

Lemma 7.8. *Recall from Lemma 5.4 that given two probability measures M, P on a common measurable space $(\Omega, \mathcal{B})$,*

$$\mathrm{var}(P, M) = 2 \sup_{F \in \mathcal{B}} |P(F) - M(F)| = 2\,\mathrm{tvar}(P, M). \tag{7.9}$$

If S is a measure for which $P \ll S$ and $M \ll S$ ($S = (P + M)/2$, for example), then also

$$\mathrm{var}(P, M) = \int |\frac{dP}{dS} - \frac{dM}{dS}| \, dS \tag{7.10}$$

and the supremum in (5.35) is achieved by the set

$$F = \{\omega : \frac{dP}{dS}(\omega) > \frac{dM}{dS}(\omega)\}.$$

Proof of Lemma 7.8: Suppose that a measure S dominating both P and M exists and define the set

$$F = \{\omega : \frac{dP}{dS}(\omega) > \frac{dM}{dS}(\omega)\}$$

and observe that

$$\begin{aligned} \int |\frac{dP}{dS} - \frac{dM}{dS}| \, dS &= \int_F (\frac{dP}{dS} - \frac{dM}{dS}) \, dS - \int_{F^c} (\frac{dP}{dS} - \frac{dM}{dS}) \, dS \\ &= P(F) - M(F) - (P(F^c) - M(F^c)) \\ &= 2(P(F) - M(F)). \end{aligned}$$

From the definition of F, however,

$$P(F) = \int_F \frac{dP}{dS} \, dS \geq \int_F \frac{dM}{dS} \, dS = M(F)$$

so that $P(F) - M(F) = |P(F) - M(F)|$. Thus we have that

$$\int |\frac{dP}{dS} - \frac{dM}{dS}| \, dS = 2|P(F) - M(F)| \leq 2 \sup_{G \in \mathcal{B}} |P(G) - M(G)| = \mathrm{var}(P, M).$$

To prove the reverse inequality, assume that $\mathcal{Q}$ approximately yields the variational distance, that is, for $\epsilon > 0$ we have

$$\sum_{Q\in\mathcal{Q}} |P(Q) - M(Q)| \geq \mathrm{var}(P, M) - \epsilon.$$

Then

$$\begin{aligned}\sum_{Q\in\mathcal{Q}} |P(Q) - M(Q)| &= \sum_{Q\in\mathcal{Q}} |\int_Q (\frac{dP}{dS} - \frac{dM}{dS})\, dS| \\ &\leq \sum_{Q\in\mathcal{Q}} \int_Q |\frac{dP}{dS} - \frac{dM}{dS}|\, dS \\ &= \int |\frac{dP}{dS} - \frac{dM}{dS}|\, dS\end{aligned}$$

which, since ϵ is arbitrary, proves that

$$\mathrm{var}(P, M) \leq \int |\frac{dP}{dS} - \frac{dM}{dS}|\, dS,$$

Combining this with the earlier inequality proves (7.10). We have already seen that this upper bound is actually achieved with the given choice of F, which completes the proof of the lemma. □

Proof of Lemma 7.7: The magnitude entropy density can be written as

$$|h(\omega)| = h(\omega) + 2h(\omega)^- \tag{7.11}$$

where $a^- = -\min(a, 0)$. This inequality immediately gives the trivial left-hand inequality of (7.7). The right-hand inequality follows from the fact that

$$\int h^- dP = \int f[\ln f]^- dM$$

and the elementary inequality $a \ln a \geq -1/e$.

The second inequality will follow from (7.11) if we can show that

$$2\int h^- dP \leq \sqrt{2D(P\|M)}.$$

Let F denote the set $\{h \leq 0\}$ and we have from (7.4) that

$$2\int h^- dP = -2\int_F h dP \leq -2P(F) \ln \frac{P(F)}{M(F)}$$

and hence using the inequality $\ln x \leq x - 1$ and Lemma 5.4

$$2\int h^- dP \leq 2P(F) \ln \frac{M(F)}{P(F)} \leq 2(M(F) - P(F))$$

$$\leq d(P,M) \leq \sqrt{2D(P\|M)},$$

completing the proof. □

Combining Lemmas 7.7 and 7.6 yields the following corollary, which generalizes Lemma 2 of [62].

Corollary 7.6. *Let P and M be two measures on a space $(\Omega, \mathcal{B})$. Suppose that $\mathcal{F}$ is a sub-σ-field and that $P_{\mathcal{F}}$ and $M_{\mathcal{F}}$ are the restrictions of P and M to $\mathcal{F}$. Assume that $M \gg P$. Define the entropy densities $h = \ln dP/dM$ and $h' = \ln dP_{\mathcal{F}}/dM_{\mathcal{F}}$. Then*

$$\int |h - h'| \, dP \leq D(P\|M) - D(P_{\mathcal{F}}\|M_{\mathcal{F}}) + \frac{2}{e}, \tag{7.12}$$

and

$$\int |h - h'| \, dP \leq$$
$$D(P\|M) - D(P_{\mathcal{F}}\|M_{\mathcal{F}}) + \sqrt{2D(P\|M) - 2D(P_{\mathcal{F}}\|M_{\mathcal{F}})}. \tag{7.13}$$

Proof. Choose the measure S as in Lemma 7.6 and then apply Lemma 7.7 with S replacing M. □

Variational Description of Divergence

As in the discrete case, divergence has a variational characterization that is a fundamental property for its applications to large deviations theory [182] [31]. We again take a detour to state and prove the property without delving into its applications.

Suppose now that P and M are two probability measures on a common probability space, say $(\Omega, \mathcal{B})$, such that $M \gg P$ and hence the density

$$f = \frac{dP}{dM}$$

is well defined. Suppose that Φ is a real-valued random variable defined on the same space, which we explicitly require to be finite-valued (it cannot assume ∞ as a value) and to have finite cumulant generating function:

$$E_M(e^{\Phi}) < \infty.$$

Then we can define a probability measure M^{Φ} by

$$M^{\Phi}(F) = \int_F \frac{e^{\Phi}}{E_M(e^{\Phi})} dM \tag{7.14}$$

and observe immediately that by construction $M \gg M^{\Phi}$ and

$$\frac{dM^{\Phi}}{dM} = \frac{e^{\Phi}}{E_M(e^{\Phi})}.$$

The measure M^{Φ} is called a "tilted" distribution. Furthermore, by construction $dM^{\Phi}/dM \neq 0$ and hence we can write

$$\int_F \frac{f}{e^{\phi}/E_M(e^{\Phi})}\, dQ = \int_F \frac{f}{e^{\phi}/E_M(e^{\Phi})} \frac{dM^{\Phi}}{dM}\, dM = \int_F f\, dM = P(F)$$

and hence $P \ll M^{\Phi}$ and

$$\frac{dP}{dM^{\Phi}} = \frac{f}{e^{\phi}/E_M(e^{\Phi})}.$$

We are now ready to state and prove the principal result of this section, a variational characterization of divergence.

Theorem 7.1. *Suppose that $M \gg P$. Then*

$$D(P\|M) = \sup_{\Phi} \left(E_P \Phi - \ln(E_M(e^{\Phi})) \right), \tag{7.15}$$

where the supremum is over all random variables Φ for which Φ is finite-valued and e^{Φ} is M-integrable.

Proof. First consider the random variable Φ defined by $\Phi = \ln f$ and observe that

$$\begin{aligned} E_P \Phi - \ln(E_M(e^{\Phi})) &= \int dP \ln f - \ln(\int dM f) \\ &= D(P\|M) - \ln \int dP = D(P\|M). \end{aligned}$$

This proves that the supremum over all Φ is no smaller than the divergence. To prove the other half observe that for any Φ,

$$H(P\|M) - \left(E_P \Phi - \ln E_M(e^{\Phi}) \right) = E_P \left(\ln \frac{dP/dM}{dP/dM^{\Phi}} \right),$$

where M^{Φ} is the tilted distribution constructed above. Since $M \gg M^{\Phi} \gg P$, we have from the chain rule for Radon-Nikodym derivatives that

$$H(P\|M) - \left(E_P \Phi - \ln E_M(e^{\Phi}) \right) = E_P \ln \frac{dP}{dM^{\Phi}} = D(P\|M^{\Phi}) \geq 0$$

from the divergence inequality, which completes the proof. Note that equality holds and the supremum is achieved if and only if $M^\Phi = P$. □

7.2 Conditional Relative Entropy

Lemmas 7.5 and 7.6 combine with basic properties of conditional probability in standard spaces to provide an alternative form of Lemma 7.6 in terms of random variables that gives an interesting connection between the densities for combinations of random variables and those for individual random variables. The results are collected in Theorem 7.2. First, however, several definitions are required. Let X and Y be random variables with standard alphabets A_X and A_Y and σ-fields $\mathcal{B}_{A_X}$ and $\mathcal{B}_{A_Y}$, respectively. Let P_{XY} and M_{XY} be two distributions on $(A_X \times A_Y, \mathcal{B}_{A_X \times A_Y})$ and assume that $M_{XY} \gg P_{XY}$. Let M_Y and P_Y denote the induced marginal distributions, e.g., $M_Y(F) = M_{XY}(A_X \times F)$. Define the (nonnegative) densities (Radon-Nikodym derivatives):

$$f_{XY} = \frac{dP_{XY}}{dM_{XY}}, f_Y = \frac{dP_Y}{dM_Y}$$

so that

$$P_{XY}(F) = \int_F f_{XY} dM_{XY};\ F \in \mathcal{B}_{A_X \times A_Y}$$
$$P_Y(F) = \int_F f_Y dM_Y;\ F \in \mathcal{B}_{A_Y}.$$

Note that $M_{XY} \gg P_{XY}$ implies that $M_Y \gg P_Y$ and hence f_Y is well defined if f_{XY} is. Define also the *conditional density*

$$f_{X|Y}(x|y) = \begin{cases} \frac{f_{XY}(x,y)}{f_Y(y)}; & \text{if } f_Y(y) > 0 \\ 1; & \text{otherwise.} \end{cases}$$

Suppose now that the entropy density

$$h_Y = \ln f_Y$$

exists and define the *conditional entropy density* or *conditional relative entropy density* by

$$h_{X|Y} = \ln f_{X|Y}.$$

Again suppose that these densities exist, we (tentatively) define the *conditional relative entropy*

$$H_{P\|M}(X|Y) = E \ln f_{X|Y} = \int dP_{XY}(x,y) \ln f_{X|Y}(x|y)$$
$$= \int dM_{XY}(x,y) f_{XY}(x,y) \ln f_{X|Y}(x|y).$$

if the expectation exists. Note that unlike unconditional relative entropies, the above definition of conditional relative entropy requires the existence of densities. Although this is sufficient in many of the applications and is convenient for the moment, it is not sufficiently general to handle all the cases we will encounter. In particular, there will be situations where we wish to define a conditional relative entropy $H_{P\|M}(X|Y)$ even though it is not true that $M_{XY} \gg P_{XY}$. Hence at the end of this section we will return to this question and provide a general definition that agrees with the current one when the appropriate densities exist and that shares those properties not requiring the existence of densities, e.g., the chain rule for relative entropy. An alternative approach to a general definition for conditional relative entropy can be found in Algoet [6].

The previous construction immediately yields the following lemma providing chain rules for densities and relative entropies.

Lemma 7.9.

$$f_{XY} = f_{X|Y} f_Y$$
$$h_{XY} = h_{X|Y} + h_Y,$$

and hence

$$D(P_{XY}\|M_{XY}) = H_{P\|M}(X|Y) + D(P_Y\|M_Y), \tag{7.16}$$

or, equivalently,

$$H_{P\|M}(X,Y) = H_{P\|M}(Y) + H_{P\|M}(X|Y), \tag{7.17}$$

a chain rule for relative entropy analogous to that for ordinary entropy. Thus if $H_{P\|M}(Y) < \infty$ so that the indeterminate form $\infty - \infty$ is avoided, then

$$H_{P\|M}(X|Y) = H_{P\|M}(X,Y) - H_{P\|M}(Y).$$

Since the alphabets are standard, there is a regular version of the conditional probabilities of X given Y under the distribution M_{XY}; that is, for each $y \in B$ there is a probability measure $M_{X|Y}(F|y)$; $F \in \mathcal{B}_A$ for fixed $F \in \mathcal{B}_{A_X}$ $M_{X|Y}(F|y)$ is a measurable function of y and such that for all $G \in \mathcal{B}_{A_Y}$

$$M_{XY}(F \times G) = E(1_G(Y) M_{X|Y}(F|Y)) = \int_G M_{X|Y}(F|y) dM_Y(y).$$

Lemma 7.10. *Given the previous definitions, define the set $\bar{B} \in \mathcal{B}_B$ to be the set of y for which*

$$\int_A f_{X|Y}(x|y)dM_{X|Y}(x|y) = 1.$$

Define $P_{X|Y}$ for $y \in \bar{B}$ by

$$P_{X|Y}(F|y) = \int_F f_{X|Y}(x|y)dM_{X|Y}(x|y);\ F \in \mathcal{B}_A$$

and let $P_{X|Y}(.|y)$ be an arbitrary fixed probability measure on $(A, \mathcal{B}_A)$ for all $y \notin \bar{B}$. Then $M_Y(\bar{B}) = 1$, $P_{X|Y}$ is a regular conditional probability for X given Y under the distribution P_{XY}, and

$$P_{X|Y} \ll M_{X|Y};\ M_Y - \text{a.e.},$$

that is, $M_Y(\{y : P_{X|Y}(\cdot|y) \ll M_{X|Y}(\cdot|y)\}) = 1$. Thus if $P_{XY} \ll M_{XY}$, we can choose regular conditional probabilities under both distributions so that with probability one under M_Y the conditional probabilities under P are dominated by those under M and

$$\frac{dP_{X|Y}}{dM_{X|Y}}(x|y) \equiv \frac{dP_{X|Y}(\cdot|y)}{dM_{X|Y}(\cdot|y)}(x) = f_{X|Y}(x|y);\ x \in A.$$

Proof. Define for each $y \in B$ the set function

$$G_y(F) = \int_F f_{X|Y}(x|y)dM_{X|Y}(x|y);\ F \in \mathcal{B}_A.$$

We shall show that $G_y(F)$, $y \in B$, $F \in \mathcal{B}_A$ is a version of a regular conditional probability of X given Y under P_{XY}. First observe using iterated expectation and the fact that conditional expectations are expectations with respect to conditional probability measures ([55], Section 5.9) that for any $F \in \mathcal{B}_B$

$$\begin{aligned}
&\int_F [\int_A f_{X|Y}(x|y)dM_{X|Y}(x|y)]\, dM_Y(y) \\
&= E(1_F(Y)E[1_A(X)f_{X|Y}|Y]) = E(1_F(Y)1_A(X)\frac{f_{XY}}{f_Y}1_{f_Y>0}) \\
&= \int 1_{A\times F}\frac{1}{f_Y}1_{\{f_Y>0\}}f_{XY}\, dM_{XY} = \int_{A\times F}\frac{1}{f_Y}1_{\{f_Y>0\}}\, dP_{XY} \\
&= \int_F \frac{1}{f_Y}1_{\{f_Y>0\}}\, dP_Y \int_F \frac{1}{f_Y}\, dP_Y,
\end{aligned}$$

where the last step follows since since the function being integrated depends only on Y and hence is measurable with respect to $\sigma(Y)$ and

therefore its expectation can be computed from the restriction of P_{XY} to $\sigma(Y)$ (see, for example, Lemma 5.3.1 of [55] or Lemma 6.3 of [58]) and since $P_Y(f_Y > 0) = 1$. We can compute this last expectation, however, using M_Y as

$$\int_F \frac{1}{f_Y} dP_Y = \int_F \frac{1}{f_Y} f_Y dM_Y = \int_F dM_Y = M_Y(F)$$

which yields finally that

$$\int_F \left[\int_A f_{X|Y}(x|y)\, dM_{X|Y}(x|y) \right] dM_Y(y) = M_Y(F); \text{ all } F \in \mathcal{B}_B.$$

If

$$\int_F g(y) dM_Y(y) = \int_F 1 dM_Y(y), \text{ all } F \in \mathcal{B}_B,$$

however, it must also be true that $g = 1$ M_Y-a.e. (See, for example, Corollary 5.3.1 of [55] or Corollary 6.1 of [58].) Thus we have M_Y-a.e. and hence also P_Y-a.e. that

$$\int_A f_{X|Y}(x|y) dM_{X|Y}(x|y)] dM_Y(y) = 1;$$

that is, $M_Y(\bar{B}) = 1$. For $y \in \bar{B}$, it follows from the basic properties of integration that G_y is a probability measure on $(A, \mathcal{B}_A)$ (see Corollary 4.4.3 of [55] or Corollary 5.4 of [58]).

By construction, $P_{X|Y}(\cdot|y) \ll M_{X|Y}(\cdot|y)$ for all $y \in \bar{B}$ and hence this is true with probability 1 under M_Y and P_Y. Furthermore, by construction

$$\frac{dP_{X|Y}(\cdot|y)}{dM_{X|Y}(\cdot|y)}(x) = f_{X|Y}(x|y).$$

To complete the proof we need only show that $P_{X|Y}$ is indeed a version of the conditional probability of X given Y under P_{XY}. To do this, fix $G \in \mathcal{B}_A$ and observe for any $F \in \mathcal{B}_B$ that

$$\begin{aligned}
\int_F P_{X|Y}(G|y)\, dP_Y(y) &= \int_F [\int_G f_{X|Y}(x|y) dM_{X|Y}(x|y)]\, dP_Y(y) \\
&= \int_F [\int_G f_{X|Y}(x|y)\, dM_{X|Y}(x|y)] f_Y(y)\, dM_Y(y) \\
&= E[1_F(Y) f_Y E[1_G(X) f_{X|Y}|Y] = E_M[1_{G\times F} f_{XY}],
\end{aligned}$$

again using iterated expectation. This immediately yields

$$\int_F P_{X|Y}(G|y)\, dP_Y(y) = \int_{G\times F} f_{XY} dM_{XY} = \int_{G\times F} dP_{XY} = P_{XY}(G \times F),$$

which proves that $P_{X|Y}(G|\mathcal{Y})$ is a version of the conditional probability of X given Y under P_{XY}, thereby completing the proof. □

Theorem 7.2. *Given the previous definitions with $M_{XY} \gg P_{XY}$, define the distribution S_{XY} by*

$$S_{XY}(F \times G) = \int_G M_{X|Y}(F|\mathcal{Y}) dP_Y(\mathcal{Y}), \tag{7.18}$$

that is, S_{XY} has P_Y as marginal distribution for Y and $M_{X|Y}$ as the conditional distribution of X given Y. Then the following statements are true:

1. $M_{XY} \gg S_{XY} \gg P_{XY}$.
2. $dS_{XY}/dM_{XY} = f_Y$ *and* $dP_{XY}/dS_{XY} = f_{X|Y}$.
3. $D(P_{XY}\|M_{XY}) = D(P_Y\|M_Y) + D(P_{XY}\|S_{XY})$, *and hence* $D(P_{XY}\|M_{XY})$ *exceeds* $D(P_Y\|M_Y)$ *by an amount* $D(P_{XY}\|S_{XY}) = H_{P\|M}(X|Y)$.

Proof. To apply Lemma 7.6 define $P = P_{XY}$, $M = M_{XY}$, $\mathcal{F} = \sigma(Y)$, $P' = P_{\sigma(Y)}$, and $M' = M_{\sigma(Y)}$. Define S by

$$S(F \times G) = \int_{F\times G} \frac{dP_{\sigma(Y)}}{dM_{\sigma(Y)}} dM_{XY},$$

for $F \in \mathcal{B}_A$ and $G \in \mathcal{B}_B$. We begin by showing that $S = S_{XY}$. All of the properties will then follow from Lemma 7.6.

For $F \in \mathcal{B}_{A_X}$ and $G \in \mathcal{B}_{A_Y}$

$$S(F \times G) = \int_{F\times G} \frac{dP_{\sigma(Y)}}{dM_{\sigma(Y)}} dM_{XY} = E\left(1_{F\times G} \frac{dP_{\sigma(Y)}}{dM_{\sigma(Y)}}\right),$$

where the expectation is with respect to M_{XY}. Using Lemma 7.5 and iterated conditional expectation (c.f. Corollary 5.9.3 of [55] or Corollary 6.5 of [58]) yields

$$\begin{aligned} E\left(1_{F\times G} \frac{dP_{\sigma(Y)}}{dM_{\sigma(Y)}}\right) &= E\left(1_F(X) 1_G(Y) \frac{dP_Y}{dM_Y}(Y)\right) \\ &= E\left(1_G(Y) \frac{dP_Y}{dM_Y}(Y) E[1_F(X)|Y]\right) \\ &= E\left(1_G(Y) \frac{dP_Y}{dM_Y}(Y) M_{X|Y}(F|Y)\right) \\ \int M_{X|Y}(F|\mathcal{Y}) \frac{dP_Y}{dM_Y}(Y) dM_Y(\mathcal{Y}) &= \int_G M_{X|Y}(F|\mathcal{Y})\, dP_Y(\mathcal{Y}), \end{aligned}$$

proving that $S = S_{XY}$. Thus Lemma 7.15 implies that $M_{XY} \gg S_{XY} \gg P_{XY}$, proving the first property.

From Lemma 7.5, $dP'/dM' = dP_{\sigma(Y)}/dM_{\sigma(Y)} = dP_Y/dM_Y = f_Y$, proving the first equality of property 2. This fact and the first property imply

the second equality of property 2 from the chain rule of Radon-Nikodym derivatives. (See, e.g., Lemma 5.7.3 of [55] or Lemma 6.6 of [58].) Alternatively, the second equality of the second property follows from Lemma 7.6 since

$$\frac{dP_{XY}}{dS_{XY}} = \frac{dP_{XY}/dM_{XY}}{dM_{XY}/dS_{XY}} = \frac{f_{XY}}{f_Y}.$$

Corollary 7.1 therefore implies that $D(P_{XY}\|M_{XY}) = D(P_{XY}\|S_{XY}) + D(S_{XY}\|M_{XY})$, which with Property 2, Lemma 7.4, and the definition of relative entropy rate imply Property 3. □

It should be observed that it is not necessarily true that $D(P_{XY}\|S_{XY}) \geq D(P_X\|M_X)$ and hence that $D(P_{XY}\|M_{XY}) \geq D(P_X\|M_X) + D(P_Y\|M_Y)$ as one might expect since in general $S_X \neq M_X$. These formulas will, however, be true in the special case where $M_{XY} = M_X \times M_Y$.

We next turn to an extension and elaboration of the theorem when there are three random variables instead of two. This will be a crucial generalization for our later considerations of processes, when the three random variables will be replaced by the current output, a finite number of previous outputs, and the infinite past.

Suppose that $M_{XYZ} \gg P_{XYZ}$ are two distributions for three standard alphabet random variables X, Y, and Z taking values in measurable spaces $(A_X, \mathcal{B}_{A_X})$, $(A_Y, \mathcal{B}_{A_Y})$, $(A_Z, \mathcal{B}_{A_Z})$, respectively. Observe that the absolute continuity implies absolute continuity for the restrictions, e.g., $M_{XY} \gg P_{XY}$ and $M_Y \gg P_Y$. Define the Radon-Nikodym derivatives f_{XYZ}, f_{YZ}, f_Y, etc. in the obvious way; for example,

$$f_{XYZ} = \frac{dP_{XYZ}}{dM_{XYZ}}.$$

Let h_{XYZ}, h_{YZ}, h_Y, etc., denote the corresponding relative entropy densities, e.g.,

$$h_{XYZ} = \ln f_{XYZ}.$$

Define as previously the conditional densities

$$f_{X|YZ} = \frac{f_{XYZ}}{f_{YZ}};\ f_{X|Y} = \frac{f_{XY}}{f_Y},$$

the conditional entropy densities

$$h_{X|YZ} = \ln f_{X|YZ};\ h_{X|Y} = \ln f_{X|Y},$$

and the conditional relative entropies

$$H_{P\|M}(X|Y) = E(\ln f_{X|Y})$$

and

$$H_{P\|M}(X|Y,Z) = E(\ln f_{X|YZ}).$$

By construction (or by double use of Lemma 7.9) we have the following chain rules for conditional relative entropy and its densities.

Lemma 7.11.

$$\begin{aligned} f_{XYZ} &= f_{X|YZ} f_{Y|Z} f_Z, \\ h_{XYZ} &= h_{X|YZ} + h_{Y|Z} + h_Z, \end{aligned}$$

and hence

$$H_{P\|M}(X,Y,Z) = H_{P\|M}(X|YZ) + H_{P\|M}(Y|Z) + H_{P\|M}(Z).$$

Corollary 7.7. *Given a distribution P_{XY}, suppose that there is a product distribution $M_{XY} = M_X \times M_Y \gg P_{XY}$. Then*

$$\begin{aligned} M_{XY} &\gg P_X \times P_Y \gg P_{XY}, \\ \frac{dP_{XY}}{d(P_X \times P_Y)} &= \frac{f_{XY}}{f_X f_Y} = \frac{f_{X|Y}}{f_X}, \\ \frac{d(P_X \times P_Y)}{dM_{XY}} &= f_X f_Y, \\ D(P_{XY}\|P_X \times P_Y) + H_{P\|M}(X) &= H_{P\|M}(X|Y), \text{ and} \\ D(P_X \times P_Y\|M_{XY}) &= H_{P\|M}(X) + H_{P\|M}(Y). \end{aligned}$$

Proof. First apply Theorem 7.2 with $M_{XY} = M_X \times M_Y$. Since M_{XY} is a product measure, $M_{X|Y} = M_X$ and $M_{XY} \gg S_{XY} = M_X \times P_Y \gg P_{XY}$ from the theorem. Next we again apply Theorem 7.2, but this time the roles of X and Y in the theorem are reversed and we replace M_{XY} in the theorem statement by the current $S_{XY} = M_X \times P_Y$ and we replace S_{XY} in the theorem statement by

$$S'_{XY}(F \times G) = \int_F S_{Y|X}(G|x)\, dP_X(x) = P_X(F)P_Y(G);$$

that is, $S'_{XY} = P_X \times P_Y$. We then conclude from the theorem that $S'_{XY} = P_X \times P_Y \gg P_{XY}$, proving the first statement. We now have that

$$M_{XY} = M_X \times M_Y \gg P_X \times P_Y \gg P_{XY}$$

and hence the chain rule for Radon-Nikodym derivatives (e.g., Lemma 5.7.3 of [55] or Lemma 6.6 of [58]) implies that

$$f_{XY} = \frac{dP_{XY}}{dM_{XY}} = \frac{dP_{XY}}{d(P_X \times P_Y)} \frac{d(P_X \times P_Y)}{d(M_X \times M_Y)}.$$

It is straightforward to verify directly that

$$\frac{d(P_X \times P_Y)}{d(M_X \times M_Y)} = \frac{dP_X}{dM_X}\frac{dP_Y}{dM_Y} = f_X f_Y$$

and hence

$$f_{XY} = \frac{dP_{XY}}{d(P_X \times P_Y)})f_X f_Y,$$

as claimed. Taking expectations using Lemma 7.4 then completes the proof (as in the proof of Corollary 7.1.) □

The lemma provides an interpretation of the product measure $P_X \times P_Y$. This measure yields independent random variables with the same marginal distributions as P_{XY}, which motivates calling $P_X \times P_Y$ the *independent approximation* or *memoryless approximation* to P_{XY}. The next corollary further enhances this name by showing that $P_X \times P_Y$ is the best such approximation in the sense of yielding the minimum divergence with respect to the original distribution.

Corollary 7.8. *Given a distribution P_{XY} let $\mathcal{M}$ denote the class of all product distributions for XY; that is, if $M_{XY} \in \mathcal{M}$, then $M_{XY} = M_X \times M_Y$. Then*

$$\inf_{M_{XY} \in \mathcal{M}} D(P_{XY} \| M_{XY}) = D(P_{XY} \| P_X \times P_Y).$$

Proof. We need only consider those M yielding finite divergence (since if there are none, both sides of the formula are infinite and the corollary is trivially true). Then

$$\begin{aligned} D(P_{XY} \| M_{XY}) &= D(P_{XY} \| P_X \times P_Y) + D(P_X \times P_Y \| M_{XY}) \\ &\geq D(P_{XY} \| P_X \times P_Y) \end{aligned}$$

with equality if and only if $D(P_X \times P_Y \| M_{XY}) = 0$, which it will be if $M_{XY} = P_X \times P_Y$. □

Recall that given random variables (X, Y, Z) with distribution M_{XYZ}, then $X \to Y \to Z$ is a Markov chain (with respect to M_{XYZ}) if for any event $F \in \mathcal{B}_{A_Z}$ with probability one

$$M_{Z|YX}(F|y, x) = M_{Z|Y}(F|y).$$

If this holds, we also say that X and Z are conditionally independent given Y. Equivalently, if we define the distribution $M_{X \times Z|Y}$ by

$$M_{X \times Z|Y}(F_X \times F_Z \times F_Y) = \int_{F_y} M_{X|Y}(F_X|y) M_{Z|Y}(F_Z|y) dM_Y(y);$$
$$F_X \in \mathcal{B}_{A_X};\ F_Z \in \mathcal{B}_{A_Z};\ F_Y \in \mathcal{B}_{A_Y};$$

then $Z \to Y \to X$ is a Markov chain if $M_{X \times Z|Y} = M_{XYZ}$. (See Section 5.10 of [55] or Section 6.10 of [58].) This construction shows that a Markov chain is symmetric in the sense that $X \to Y \to Z$ if and only if $Z \to Y \to X$.

Note that for any measure M_{XYZ}, $X \to Y \to Z$ is a Markov chain under $M_{X\times Z|Y}$ by construction.

The following corollary highlights special properties of the various densities and relative entropies when the dominating measure is a Markov chain. It will lead to the idea of a Markov approximation to an arbitrary distribution on triples extending the independent approximation of the previous corollary.

Corollary 7.9. *Given a probability space, suppose that $M_{XYZ} \gg P_{XYZ}$ are two distributions for a random vector (X, Y, Z) with the property that $Z \to Y \to X$ forms a Markov chain under M. Then*

$$M_{XYZ} \gg P_{X\times Z|Y} \gg P_{XYZ}$$

and

$$\frac{dP_{XYZ}}{dP_{X\times Z|Y}} = \frac{f_{X|YZ}}{f_{X|Y}} \tag{7.19}$$

$$\frac{dP_{X\times Z|Y}}{dM_{XYZ}} = f_{YZ} f_{X|Y}. \tag{7.20}$$

Thus

$$\ln \frac{dP_{XYZ}}{dP_{X\times Z|Y}} + h_{X|Y} = h_{X|YZ}$$

$$\ln \frac{dP_{X\times Z|Y}}{dM_{XYZ}} = h_{YZ} + h_{X|Y}$$

and taking expectations yields

$$D(P_{XYZ}\|P_{X\times Z|Y}) + H_{P\|M}(X|Y) = H_{P\|M}(X|YZ)$$

$$D(P_{X\times Z|Y}\|M_{XYZ}) = D(P_{YZ}\|M_{YZ}) + H_{P\|M}(X|Y).$$

Furthermore,

$$P_{X\times Z|Y} = \overline{P_{X|Y} P_{YZ}}, \tag{7.21}$$

that is,

$$P_{X\times Z|Y}(F_X \times F_Z \times F_Y) = \int_{F_Y\times F_Z} P_{X|Y}(F_X|y)\, dP_{ZY}(z, y). \tag{7.22}$$

Lastly, if $Z \to Y \to X$ is a Markov chain under M, then it is also a Markov chain under P if and only if

$$h_{X|Y} = h_{X|YZ} \tag{7.23}$$

in which case

$$H_{P\|M}(X|Y) = H_{P\|M}(X|YZ). \tag{7.24}$$

Proof. Define

$$g(x,y,z) = \frac{f_{X|YZ}(x|y,z)}{f_{X|Y}(x|y)} = \frac{f_{XYZ}(x,y,z)}{f_{YZ}(y,z)} \frac{f_Y(y)}{f_{XY}(x,y)}$$

and simplify notation by defining the measure $Q = P_{X\times Z|Y}$. Note that $Z \to Y \to X$ is a Markov chain with respect to Q. To prove the first statement of the corollary requires proving the following relation:

$$P_{XYZ}(F_X \times F_Y \times F_Z) = \int_{F_X\times F_Y\times F_Z} g\,dQ;$$
$$\text{all } F_X \in \mathcal{B}_{A_X}, F_Z \in \mathcal{B}_{A_Z}, F_Y \in \mathcal{B}_{A_Y}.$$

From iterated expectation with respect to Q (e.g., Section 5.9 of [55] or Section 6.9 of [58])

$$\begin{aligned} E(g1_{F_X}(X)1_{F_Z}(Z)1_{F_Y}(Y)) &= E(1_{F_Y}(Y)1_{F_Z}(Z)E(g1_{F_X}(X)|YZ)) \\ &= \int 1_{F_Y}(y)1_{F_Z}(z)\Big(\int_{F_X} g(x,y,z)\,dQ_{X|YZ}(x|y,z)\Big)\,dQ_{YZ}(y,z). \end{aligned}$$

Since $Q_{YZ} = P_{YZ}$ and $Q_{X|YZ} = P_{X|Y}$ Q-a.e. by construction, the previous formula implies that

$$\int_{F_X\times F_Y\times F_Z} g\,dQ = \int_{F_Y\times F_Z} dP_{YZ} \int_{F_X} g\,dP_{X|Y}.$$

This proves (7.21. Since $M_{XYZ} \gg P_{XYZ}$, we also have that $M_{XY} \gg P_{XY}$ and hence application of Theorem 7.2 yields

$$\begin{aligned} \int_{F_X\times F_Y\times F_Z} g\,dQ &= \int_{F_Y\times F_Z} dP_{YZ} \int_{F_X} g f_{X|Y}\,dM_{X|Y} \\ &= \int_{F_Y\times F_Z} dP_{YZ} \int_{F_X} f_{X|YZ}\,dM_{X|Y}. \end{aligned}$$

By assumption, however, $M_{X|Y} = M_{X|YZ}$ a.e. and therefore

$$\begin{aligned} \int_{F_X\times F_Y\times F_Z} g\,dQ &= \int_{F_Y\times F_Z} dP_{YZ} \int_{F_X} f_{X|YZ}\,dM_{X|YZ} \\ &= \int_{F_Y\times F_Z} dP_{YZ} \int_{F_X} dP_{X|YZ} \\ &= P_{XYZ}(F_X \times F_Y \times F_Z), \end{aligned}$$

where the final step follows from iterated expectation. This proves (7.19) and that $Q \gg P_{XYZ}$.

To prove (7.20) we proceed in a similar manner and replace g by $f_{X|Y}f_{ZY}$ and replace Q by $M_{XYZ} = M_{X\times Y|Z}$. Also abbreviate $P_{X\times Y|Z}$ to

$\hat{P}$. As in the proof of (7.19) we have since $Z \to Y \to X$ is a Markov chain under M that

$$\begin{aligned}\int_{F_X \times F_Y \times F_Z} g\, dQ &= \int_{F_Y \times F_Z} dM_{YZ} \int_{F_X} g\, dM_{X|Y} \\ &= \int_{F_Y \times F_Z} f_{ZY}\, dM_{YZ} \left(\int_{F_X} f_{X|Y}\, dM_{X|Y} \right) \\ &= \int_{F_Y \times F_Z} dP_{YZ} \left(\int_{F_X} f_{X|Y}\, dM_{X|Y} \right).\end{aligned}$$

From Theorem 7.2 this is

$$\int_{F_Y \times F_Z} P_{X|Y}(F_X|y)\, dP_{YZ}.$$

But $P_{YZ} = \hat{P}_{YZ}$ and

$$P_{X|Y}(F_X|y) = \hat{P}_{X|Y}(F_X|y) = \hat{P}_{X|YZ}(F_X|yz)$$

since $\hat{P}$ yields a Markov chain. Thus the previous formula is $\hat{P}(F_X \times F_Y \times F_Z)$, proving (7.20) and the corresponding absolute continuity.

If $Z \to Y \to X$ is a Markov chain under both M and P, then $P_{X \times Z|Y} = P_{XYZ}$ and hence

$$\frac{dP_{XYZ}}{dP_{X \times Z|Y}} = 1 = \frac{f_{X|YZ}}{f_{X|Y}},$$

which implies (7.23). Conversely, if (7.23) holds, then $f_{X|YZ} = f_{X|Y}$ which with (7.19) implies that $P_{XYZ} = P_{X \times Z|Y}$, proving that $Z \to Y \to X$ is a Markov chain under P. □

The previous corollary and one of the constructions used will prove important later and hence it is emphasized now with a definition and another corollary giving an interesting interpretation.

Given a distribution P_{XYZ}, define the distribution $P_{X \times Z|Y}$ as the *Markov approximation* to P_{XYZ}. Abbreviate $P_{X \times Z|Y}$ to $\hat{P}$. The definition has two motivations. First, the distribution $\hat{P}$ makes $Z \to Y \to X$ a Markov chain which has the same initial distribution $\hat{P}_{ZY} = P_{ZY}$ and the same conditional distribution $\hat{P}_{X|Y} = P_{X|Y}$, the only difference is that $\hat{P}$ yields a Markov chain, that is, $\hat{P}_{X|ZY} = \hat{P}_{X|Y}$. The second motivation is the following corollary which shows that of all Markov distributions, $\hat{P}$ is the closest to P in the sense of minimizing the divergence.

Corollary 7.10. *Given a distribution $P = P_{XYZ}$, let $\mathcal{M}$ denote the class of all distributions for XYZ for which $Z \to Y \to X$ is a Markov chain under M_{XYZ} ($M_{XYZ} = M_{X \times Z|Y}$). Then*

$$\inf_{M_{XYZ} \in \mathcal{M}} D(P_{XYZ} \| M_{XYZ}) = D(P_{XYZ} \| P_{X \times Z|Y});$$

that is, the infimum is a minimum and it is achieved by the Markov approximation.

Proof. If no M_{XYZ} in the constraint set satisfies $M_{XYZ} \gg P_{XYZ}$, then both sides of the above equation are infinite. Hence confine interest to the case $M_{XYZ} \gg P_{XYZ}$. Similarly, if all such M_{XYZ} yield an infinite divergence, we are done. Hence we also consider only M_{XYZ} yielding finite divergence. Then the previous corollary implies that $M_{XYZ} \gg P_{X\times Z|Y} \gg P_{XYZ}$ and hence

$$\begin{aligned} D(P_{XYZ}\|M_{XYZ}) &= D(P_{XYZ}\|P_{X\times Z|Y}) + D(P_{X\times Z|Y}\|M_{XYZ}) \\ &\geq D(P_{XYZ}\|P_{X\times Z|Y}) \end{aligned}$$

with equality if and only if

$$D(P_{X\times Z|Y}\|M_{XYZ}) = D(P_{YZ}\|M_{YZ}) + H_{P\|M}(X|Y) = 0.$$

But this will be zero if M is the Markov approximation to P since then $M_{YZ} = P_{YZ}$ and $M_{X|Y} = P_{X|Y}$ by construction. □

Generalized Conditional Relative Entropy

We now return to the issue of providing a general definition of conditional relative entropy, that is, one which does not require the existence of the densities or, equivalently, the absolute continuity of the underlying measures. We require, however, that the general definition reduce to that considered thus far when the densities exist so that all of the results of this section will remain valid when applicable. The general definition takes advantage of the basic construction of the early part of this section. Once again let M_{XY} and P_{XY} be two measures, where we no longer assume that $M_{XY} \gg P_{XY}$. Define as in Theorem 7.2 the modified measure S_{XY} by

$$S_{XY}(F \times G) = \int_G M_{X|Y}(F|y)\,dP_Y(y); \tag{7.25}$$

that is, S_{XY} has the same Y marginal as P_{XY} and the same conditional distribution of X given Y as M_{XY}. We now replace the previous definition by the following: The *conditional relative entropy* is defined by

$$H_{P\|M}(X|Y) = D(P_{XY}\|S_{XY}). \tag{7.26}$$

If $M_{XY} \gg P_{XY}$ as before, then from Theorem 7.2 this is the same quantity as the original definition and there is no change. The divergence of (7.26), however, is well-defined even if it is not true that $M_{XY} \gg P_{XY}$ and hence the densities used in the original definition do not work. The key

question is whether or not the chain rule

$$H_{P\|M}(Y) + H_{P\|M}(X|Y) = H_{P\|M}(XY) \tag{7.27}$$

remains valid in the more general setting. It has already been proven in the case that $M_{XY} \gg P_{XY}$, hence suppose this does not hold. In this case, if it is also true that $M_Y \gg P_Y$ does not hold, then both the marginal and joint relative entropies will be infinite and (7.27) again must hold since the conditional relative entropy is nonnegative. Thus we need only show that the formula holds for the case where $M_Y \gg P_Y$ but it is not true that $M_{XY} \gg P_{XY}$. By assumption there must be an event F for which

$$M_{XY}(F) = \int M_{X|Y}(F_y)\, dM_Y(y) = 0$$

but

$$P_{XY}(F) = \int P_{X|Y}(F_y)\, dP_Y(y) \neq 0,$$

where $F_y = \{(x, y) : (x, y) \in F\}$ is the section of F at F_y. Thus $M_{X|Y}(F_y) = 0$ M_Y-a.e. and hence also P_Y-a.e. since $M_Y \gg P_Y$. Thus

$$S_{XY}(F) = \int M_{X|Y}(F_y)\, dP_Y(y) = 0$$

and hence it is not true that $S_{XY} \gg P_{XY}$ and therefore

$$D(P_{XY}\|S_{XY}) = \infty,$$

which proves that the chain rule holds in the general case.

It can happen that P_{XY} is not absolutely continuous with respect to M_{XY}, and yet $D(P_{XY}\|S_{XY}) < \infty$ and hence $P_{XY} \ll S_{XY}$ and hence

$$H_{P\|M}(X|Y) = \int dP_{XY} \ln \frac{dP_{XY}}{dS_{XY}},$$

in which case it makes sense to define the conditional density

$$f_{X|Y} \equiv \frac{dP_{XY}}{dS_{XY}}$$

so that exactly as in the original tentative definition in terms of densities (7.16) we have that

$$H_{P\|M}(X|Y) = \int dP_{XY} \ln f_{X|Y}.$$

Note that this allows us to define a meaningful conditional density even though the joint density f_{XY} does not exist! If the joint density does ex-

ist, then the conditional density reduces to the previous definition from Theorem 7.2.

We summarize the generalization in the following theorem.

Theorem 7.3. *The conditional relative entropy defined by (7.26) and (7.25) agrees with the definition (7.16) in terms of densities and satisfies the chain rule (7.27). If the conditional relative entropy is finite, then*

$$H_{P\|M}(X|Y) = \int dP_{XY} \ln f_{X|Y},$$

where the conditional density is defined by

$$f_{X|Y} \equiv \frac{dP_{XY}}{dS_{XY}}.$$

If $M_{XY} \gg P_{XY}$, then this reduces to the usual definition

$$f_{X|Y} = \frac{f_{XY}}{f_Y}.$$

The generalizations can be extended to three or more random variables in the obvious manner.

7.3 Limiting Entropy Densities

We now combine several of the results of the previous section to obtain results characterizing the limits of certain relative entropy densities.

Lemma 7.12. *Given a probability space $(\Omega, \mathcal{B})$ and an asymptotically generating sequence of sub-σ-fields $\mathcal{F}_n$ and two measures $M \gg P$, let $P_n = P_{\mathcal{F}_n}$, $M_n = M_{\mathcal{F}_n}$ and let $h_n = \ln dP_n/dM_n$ and $h = \ln dP/dM$ denote the entropy densities. If $D(P\|M) < \infty$, then*

$$\lim_{n\to\infty} \int |h_n - h|\, dP = 0,$$

that is, $h_n \to h$ in L^1. Thus the entropy densities h_n are uniformly integrable.

Proof. Follows from the Corollaries 7.3 and 7.6. □

The following lemma is Lemma 1 of Algoet and Cover [7].

Lemma 7.13. *Given a sequence of nonnegative random variables $\{f_n\}$ defined on a probability space $(\Omega, \mathcal{B}, P)$, suppose that*

$$E(f_n) \le 1; \quad \textit{all } n.$$

Then

$$\limsup_{n\to\infty} \frac{1}{n} \ln f_n \le 0.$$

Proof. Given any $\epsilon > 0$ the Markov inequality and the given assumption imply that

$$P(f_n > e^{n\epsilon}) \le \frac{E(f_n)}{e^{n\epsilon}} \le e^{-n\epsilon}.$$

We therefore have that

$$P(\frac{1}{n} \ln f_n \ge \epsilon) \le e^{-n\epsilon}$$

and therefore

$$\sum_{n=1}^{\infty} P(\frac{1}{n} \ln f_n \ge \epsilon) \le \sum_{n=1}^{\infty} e^{-n\epsilon} = \frac{1}{e^{\epsilon-1}} < \infty,$$

Thus from the Borel-Cantelli lemma (Lemma 4.6.3 of [55] or Lemma 5.17 of [58]), $P(n^{-1}h_n \ge \epsilon \text{ i.o.}) = 0$. Since ϵ is arbitrary, the lemma is proved. □

The lemma easily gives the first half of the following result, which is also due to Algoet and Cover [7], but the proof is different here and does not use martingale theory. The result is the generalization of Lemma 3.19.

Theorem 7.4. *Given a probability space $(\Omega, \mathcal{B})$ and an asymptotically generating sequence of sub-σ-fields $\mathcal{F}_n$, let M and P be two probability measures with their restrictions $M_n = M_{\mathcal{F}_n}$ and $P_n = P_{\mathcal{F}_n}$. Suppose that $M_n \gg P_n$ for all n and define $f_n = dP_n/dM_n$ and $h_n = \ln f_n$. Then*

$$\limsup_{n\to\infty} \frac{1}{n} h_n \le 0, M - \text{a.e.}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} h_n \ge 0, P - \text{a.e.}.$$

If it is also true that $M \gg P$ (e.g., $D(P\|M) < \infty$), then

$$\lim_{n\to\infty} \frac{1}{n} h_n = 0, P - \text{a.e.}.$$

Proof. Since

$$E_M f_n = E_{M_n} f_n = 1,$$

the first statement follows from the previous lemma. To prove the second statement consider the probability

$$P(-\frac{1}{n}\ln\frac{dP_n}{M_n} > \epsilon) = P_n(-\frac{1}{n}\ln f_n > \epsilon) = P_n(f_n < e^{-n\epsilon})$$
$$= \int_{f_n<e^{-n\epsilon}} dP_n = \int_{f_n<e^{-n\epsilon}} f_n\, dM_n$$
$$< e^{-n\epsilon}\int_{f_n<e^{-n\epsilon}} dM_n = e^{-n\epsilon}M_n(f_n < e^{-n\epsilon}) \le e^{-n\epsilon}.$$

Thus it has been shown that

$$P(\frac{1}{n}h_n < -\epsilon) \le e^{-n\epsilon}$$

and hence again applying the Borel-Cantelli lemma we have that

$$P(n^{-1}h_n \le -\epsilon \text{ i.o.}) = 0$$

which proves the second claim of the theorem.

If $M \gg P$, then the first result also holds P-a.e., which with the second result proves the final claim. □

Barron [8] provides an additional property of the sequence h_n/n. If $M \gg P$, then the sequence h_n/n is dominated by an integrable function.

7.4 Information for General Alphabets

We can now use the divergence results of the previous sections to generalize the definitions of information and to develop their basic properties. We assume now that all random variables and processes are defined on a common underlying probability space $(\Omega, \mathcal{B}, P)$. As we have seen how all of the various information quantities–entropy, mutual information, conditional mutual information–can be expressed in terms of divergence in the finite case, we immediately have definitions for the general case. Given two random variables X and Y, define the average mutual information between them by

$$I(X;Y) = D(P_{XY}\|P_X \times P_Y), \tag{7.28}$$

where P_{XY} is the joint distribution of the random variables X and Y and $P_X \times P_Y$ is the product distribution.

Define the entropy of a single random variable X by

$$H(X) = I(X;X). \tag{7.29}$$

From the definition of divergence this implies that

$$I(X;Y) = \sup_{\mathcal{Q}} H_{P_{XY}\|P_X\times P_Y}(\mathcal{Q}).$$

From Dobrushin's theorem (Lemma 7.3), the supremum can be taken over partitions whose elements are contained in generating field. Letting the generating field be the field of all rectangles of the form $F \times G$, $F \in \mathcal{B}_{A_X}$ and $G \in \mathcal{B}_{A_Y}$, we have the following lemma which is often used as a definition for mutual information.

Lemma 7.14.

$$I(X;Y) = \sup_{q,r} I(q(X);r(Y)),$$

where the supremum is over all quantizers q and r of A_X and A_Y. Hence there exist sequences of increasingly fine quantizers $q_n : A_X \to A_n$ and $r_n : A_Y \to B_n$ such that

$$I(X;Y) = \lim_{n\to\infty} I(q_n(X); r_n(Y)).$$

Applying this result to entropy we have that

$$H(X) = \sup_{q} H(q(X)),$$

where the supremum is over all quantizers.

By "increasingly fine" quantizers is meant that the corresponding partitions $\mathcal{Q}_n = \{q_n^{-1}(a);\ a \in A_n\}$ are successive refinements, e.g., atoms in $\mathcal{Q}_n$ are unions of atoms in $\mathcal{Q}_{n+1}$. (If this were not so, a new quantizer could be defined for which it was true.) There is an important drawback to the lemma (which will shortly be removed in Lemma 7.18 for the special case where the alphabets are standard): the quantizers which approach the suprema may depend on the underlying measure P_{XY}. In particular, a sequence of quantizers which work for one measure need not work for another.

An immediate corollary of Lemma 7.14 extends an inequality known for the finite case to general alphabets. It is useful when one of the random variables has finite entropy.

Corollary 7.11.

$$I(X;Y) \le H(Y).$$

Proof. Let quantizers q and r be quantizers of X and Y as previously. The finite alphabet result of Lemma 3.11 implies that $I(q(X);r(Y)) \le H(r(Y))$. Taking the supremum over the quantizers yields the corollary. □

Given a third random variable Z, let A_X, A_Y, and A_Z denote the alphabets of X, Y, and Z and define the conditional average mutual information

$$I(X;Y|Z) = D(P_{XYZ}\|P_{X\times Y|Z}). \tag{7.30}$$

This is the extension of the discrete alphabet definition of (3.27) and it makes sense only if the distribution $P_{X\times Y|Z}$ exists, which is the case if the alphabets are standard but may not be the case otherwise. We shall later provide an alternative definition due to Wyner [197] that is valid more generally and equal to the above when the spaces are standard.

Note that $I(X;Y|Z)$ can be interpreted using Corollary 7.10 as the divergence between P_{XYZ} and its Markov approximation.

Combining these definitions with Lemma 7.1 yields the following generalizations of the discrete alphabet results.

Lemma 7.15. *Given two random variables X and Y, then*

$$I(X;Y) \ge 0$$

with equality if and only if X and Y are independent. Given three random variables X, Y, and Z, then

$$I(X;Y|Z) \ge 0$$

with equality if and only if $Y \to Z \to X$ form a Markov chain.

Proof. The first statement follow from Lemma 7.1 since X and Y are independent if and only if $P_{XY} = P_X \times P_Y$. The second statement follows from (7.30) and the fact that $Y \to Z \to X$ is a Markov chain if and only if $P_{XYZ} = P_{X\times Y|Z}$ (see, e.g., Corollary 5.10.1 of [55] or Corollary 6.7 of [58]). □

The properties of divergence provide means of computing and approximating these information measures. From Lemma 7.4, if $I(X;Y)$ is finite, then

$$I(X;Y) = \int \ln \frac{dP_{XY}}{d(P_X \times P_Y)}\, dP_{XY} \tag{7.31}$$

and if $I(X;Y|Z)$ is finite, then

$$I(X;Y|Z) = \int \ln \frac{dP_{XYZ}}{dP_{X\times Y|Z}}\, dP_{XYZ}. \tag{7.32}$$

For example, if X, Y are two random variables whose distribution is absolutely continuous with respect to Lebesgue measure $dxdy$ and hence which have a pdf $f_{XY}(x,y) = dP_{XY}(xy)/dxdy$, then

$$I(X;Y) = \int dxdy f_{XY}(xy) \ln \frac{f_{XY}(x,y)}{f_X(x) f_Y(y)},$$

where f_X and f_Y are the marginal pdf's, e.g.,

$$f_X(x) = \int f_{XY}(x, y)\, dy = \frac{dP_X(x)}{dx}.$$

In the cases where these densities exist, we define the information densities

$$i_{X;Y} = \ln \frac{dP_{XY}}{d(P_X \times P_Y)}$$

$$i_{X;Y|Z} = \ln \frac{dP_{XYZ}}{dP_{X\times Y|Z}}. \tag{7.33}$$

The results of Section 7.2 can be used to provide conditions under which the various information densities exist and to relate them to each other. Corollaries 7.7 and 7.8 combined with the definition of mutual information immediately yield the following two results.

Lemma 7.16. *Let X and Y be standard alphabet random variables with distribution P_{XY}. Suppose that there exists a product distribution $M_{XY} = M_X \times M_Y$ such that $M_{XY} \gg P_{XY}$. Then*

$$M_{XY} \gg P_X \times P_Y \gg P_{XY},$$

$$i_{X;Y} = \ln(f_{XY}/f_X f_Y) = \ln(f_{X|Y}/f_X)$$

and

$$I(X;Y) + H_{P\|M}(X) = H_{P\|M}(X|Y). \tag{7.34}$$

Comment: This generalizes the fact that $I(X;Y) = H(X) - H(X|Y)$ for the finite alphabet case. The sign reversal results from the difference in definitions of relative entropy and entropy. Note that this implies that unlike ordinary entropy, relative entropy is *increased* by conditioning, at least when the reference measure is a product measure.

The previous lemma provides an apparently more general test for the existence of a mutual information density than the requirement that $P_X \times P_Y \gg P_{XY}$, it states that if P_{XY} is dominated by *any* product measure, then it is also dominated by the product of its own marginals and hence the densities exist. The generality is only apparent, however, as the given condition implies from Corollary 7.7 that the distribution is dominated by its independent approximation. Restating Corollary 7.7 in terms of mutual information yields the following.

Corollary 7.12. *Given a distribution P_{XY} let $\mathcal{M}$ denote the collection of all product distributions $M_{XY} = M_X \times M_Y$. Then*

$$I(X;Y) = \inf_{M_{XY}\in\mathcal{M}} H_{P\|M}(X|Y) = \inf_{M_{XY}\in\mathcal{M}} D(P_{XY}\|M_{XY}).$$

The next result is an extension of Lemma 7.16 to conditional information densities and relative entropy densities when three random variables are considered. It follows immediately from Corollary 7.9 and the definition of conditional information density.

Lemma 7.17. *(The chain rule for relative entropy densities) Suppose that $M_{XYZ} \gg P_{XYZ}$ are two distributions for three standard alphabet random variables and that $Z \to Y \to X$ is a Markov chain under M_{XYZ}. Let $f_{X|YZ}$, $f_{X|Y}$, $h_{X|YZ}$, and $h_{X|Y}$ be as in Section 7.2. Then $P_{X\times Z|Y} \gg P_{XYZ}$,*

$$h_{X|YZ} = i_{X;Z|Y} + h_{X|Y} \tag{7.35}$$

and

$$H_{P\|M}(X|Y,Z) = I(X;Z|Y) + H_{P\|M}(X|Y). \tag{7.36}$$

Thus, for example,

$$H_{P\|M}(X|Y,Z) \geq H_{P\|M}(X|Y).$$

As with Corollary 7.12, the lemma implies a variational description of conditional mutual information. The result is just a restatement of Corollary 7.10.

Corollary 7.13. *Given a distribution P_{XYZ} let $\mathcal{M}$ denote the class of all distributions for XYZ under which $Z \to Y \to X$ is a Markov chain, then*

$$I(X;Z|Y) = \inf_{M_{XYZ}\in\mathcal{M}} H_{P\|M}(X|Y,Z) = \inf_{M_{XYZ}\in\mathcal{M}} D(P_{XYZ}\|M_{XYZ}),$$

and the minimum is achieved by $M_{XYZ} = P_{X\times Z|Y}$.

The following corollary relates the information densities of the various information measures and extends Kolmogorov's equality to standard alphabets.

Corollary 7.14. *(The chain rule for information densities and Kolmogorov's formula.) Suppose that X,Y, and Z are random variables with standard alphabets and distribution P_{XYZ}. Suppose also that there exists a distribution $M_{XYZ} = M_X \times M_{YZ}$ such that $M_{XYZ} \gg P_{XYZ}$. (This is true, for example, if $P_X \times P_{YZ} \gg P_{XYZ}$.) Then the information densities $i_{X;Z|Y}$, $i_{X;Y}$, and $i_{X;(YZ)}$ exist and are related by*

$$i_{X;Z|Y} + i_{X;Y} = i_{X;(Y,Z)} \tag{7.37}$$

and

$$I(X;Z|Y) + I(X;Y) = I(X;(Y,Z)). \tag{7.38}$$

Proof. If $M_{XYZ} = M_X \times M_{YZ}$, then $Z \to Y \to X$ is trivially a Markov chain since $M_{X|YZ} = M_{X|Y} = M_X$. Thus the previous lemma can be applied to this M_{XYZ} to conclude that $P_{X\times Z|Y} \gg P_{XYZ}$ and that (7.35) holds. We also have that $M_{XY} = M_X \times M_Y \gg P_{XY}$. Thus all of the densities exist. Applying Lemma 7.16 to the product measures $M_{XY} = M_X \times M_Y$ and $M_{X(YZ)} = M_X \times M_{YZ}$ in (7.35) yields

$$\begin{aligned} i_{X;Z|Y} &= h_{X|YZ} - h_{X|Y} = \ln f_{X|YZ} - \ln f_{X|Y} \\ &= \ln \frac{f_{X|YZ}}{f_X} - \ln \frac{f_{X|Y}}{f_X} = i_{X;YZ} - i_{X;Y}. \end{aligned}$$

Taking expectations completes the proof. □

The previous corollary implies that if $P_X \times P_{YZ} \gg P_{XYZ}$, then also $P_{X\times Z|Y} \gg P_{XYZ}$ and $P_X \times P_Y \gg P_{XY}$ and hence that the existence of $i_{X;(Y,Z)}$ implies that of $i_{X;Z|Y}$ and $i_{X;Y}$. The following result provides a converse to this fact: the existence of the latter two densities implies that of the first. The result is due to Dobrushin [32]. (See also Theorem 3.6.1 of Pinsker [150] and the translator's comments.)

Corollary 7.15. *If $P_{X\times Z|Y} \gg P_{XYZ}$ and $P_X \times P_Y \gg P_{XY}$, then also $P_X \times P_{YZ} \gg P_{XYZ}$ and*

$$\frac{dP_{XYZ}}{d(P_X \times P_{YZ})} = \frac{dP_{XY}}{d(P_X \times P_Y)}.$$

Thus the conclusions of Corollary 7.14 hold.

Proof. The key to the proof is the demonstration that

$$\frac{dP_{XY}}{d(P_X \times P_Y)} = \frac{dP_{X\times Z|Y}}{d(P_X \times P_{YZ})}, \tag{7.39}$$

which implies that $P_X \times P_{YZ} \gg P_{X\times Z|Y}$. Since it is assumed that $P_{X\times Z|Y} \gg P_{XYZ}$, the result then follows from the chain rule for Radon-Nikodym derivatives.

Eq. (7.39) will be proved if it is shown that for all $F_X \in \mathcal{B}_{A_X}$, $F_Y \in \mathcal{B}_{A_Y}$, and $F_Z \in \mathcal{B}_{A_Z}$,

$$P_{X\times Z|Y}(F_X \times F_Z \times F_Y) = \int_{F_X\times F_Z\times F_Y} \frac{dP_{XY}}{d(P_X \times P_Y)} d(P_X \times P_{YZ}). \tag{7.40}$$

The thrust of the proof is the demonstration that for any measurable nonnegative function $f(x, z)$

$$\int_{z\in F_Z} f(x,y)\, d(P_X\times P_{YZ})(x,y,z) = \int f(x,y) P_{Z|Y}(F_Z|y) d(P_X\times P_Y)(x,y). \tag{7.41}$$

The lemma will then follow by substituting

$$f(x,y) = \frac{dP_{XY}}{d(P_X \times P_Y)}(x,y)1_{F_X}(x)1_{F_Y}(y)$$

into (7.41) to obtain (7.40).

To prove (7.41) first consider indicator functions of rectangles: $f(x,y) = 1_{F_X}(x)1_{F_Y}(y)$. Then both sides of (7.41) equal $P_X(F_X)P_{YZ}(F_Y \times F_Y)$ from the definitions of conditional probability and product measures. In particular, from Lemma 5.10.1 of [55] or Corollary 6.7 of [58] the left-hand side is

$$\begin{aligned}\int_{z\in F_Z} 1_{F_X}(x)1_{F_Y}(y)\, d(P_X \times P_{YZ})(x,y,z) &= (\int 1_{F_X} dP_X)(\int 1_{F_Y\times F_Z}\, dP_{YZ}) \\ &= P_X(F)P_{YZ}(F_Y \times F_Z)\end{aligned}$$

and the right-hand side is

$$\int 1_{F_X}(x)1_{F_Y}(y)P_{Z|Y}(F_Z|y)\, d(P_X \times P_Y)(x,y) =$$

$$(\int 1_{F_X}(x)\, dP_X(x))(\int 1_{F_Y}(y)P_{Z|Y}(F_Z|y)\, dP_Y(y)) = P_X(F)P_{YZ}(F_Y \times F_Z),$$

as claimed. This implies (7.41) holds also for simple functions and hence also for positive functions by the usual approximation arguments. □

Note that Kolmogorov's formula (7.36) gives a formula for computing conditional mutual information as

$$I(X;Z|Y) = I(X;(Y,Z)) - I(X;Y).$$

The formula is only useful if it is not indeterminate, that is, not of the form $\infty - \infty$. This will be the case if $I(Y;Z)$ (the smaller of the two mutual informations) is finite.

Corollary 7.5 provides a means of approximating mutual information by that of finite alphabet random variables. Assume now that the random variables X, Y have standard alphabets. For, say, random variable X with alphabet A_X there must then be an asymptotically generating sequence of finite fields $\mathcal{F}_X(n)$ with atoms $\mathcal{A}_X(n)$, that is, all of the members of $\mathcal{F}_X(n)$ can be written as unions of disjoint sets in $\mathcal{A}_X(n)$ and $\mathcal{F}_X(n) \uparrow \mathcal{B}_{A_X}$; that is, $\mathcal{B}_{A_X} = \sigma(\bigcup_n \mathcal{F}_X(n))$. The atoms $\mathcal{A}_X(n)$ form a partition of the alphabet of X.

Consider the divergence result of Corollary 7.5. with $P = P_{XY}$, $M = P_X \times P_Y$ and quantizer $q^{(n)}(x,y) = (q_X^{(n)}(x), q_Y^{(n)}(y))$. Consider the limit $n \to \infty$. Since $\mathcal{F}_X(n)$ asymptotically generates $\mathcal{B}_{A_X}$ and $\mathcal{F}_Y(n)$ asymptotically generates $\mathcal{B}_{A_Y}$ and since the pair σ-field $\mathcal{B}_{A_X\times A_Y}$ is generated by rectangles, the field generated by all sets of the form $F_X \times F_Y$ with $F_X \in \mathcal{F}_X(n)$, some n, and $F_Y \in \mathcal{F}_Y(m)$, some m, generates $\mathcal{B}_{A_X\times A_Y}$. Hence Corollary 7.5 yields the first result of the following lemma. The

second is a special case of the first. The result shows that the increasingly fine quantizers of Lemma 7.14 can be chosen in a manner not depending on the underlying measure if the alphabets are standard.

Lemma 7.18. *Suppose that X and Y are random variables with standard alphabets defined on a common probability space. Suppose that $q_X^{(n)}$, $n = 1, 2, \cdots$ is a sequence of quantizers for A_X such that the corresponding partitions asymptotically generate $\mathcal{B}_{A_X}$. Define quantizers for Y similarly. Then for any distribution P_{XY}*

$$I(X;Y) = \lim_{n\to\infty} I(q_X^{(n)}(X); q_Y^{(n)}(Y))$$

and

$$H(X) = \lim_{n\to\infty} H(q_X^{(n)}(X));$$

that is, the same quantizer sequence works for all distributions.

The following lemma generalizes Lemma 3.14 to standard alphabets. The concavity with respect to the source follows in a manner similar to the entropy result of Lemma 7.18 by combining the finite-alphabet result of Lemma 3.14 with limiting quantization. The convexity with respect to the channel does not readily follow in the same way because a channel can not be quantized without using an input distribution to form a joint distribution. The proof instead mimics the corresponding proof of the finite case based on the convexity of divergence of Lemma 7.2, which in turn follows from the finite alphabet result.

Lemma 7.19. *Let X and Y be random variables with standard alphabets A_X and A_Y. Let μ denote a distribution on $(A_X, \mathcal{B}_{A_X})$, and let ν be a regular conditional distribution $\nu(F|x) = Pr(Y \in F|X = x)$, $x \in A_X$, $F \in \mathcal{B}_{A_Y}$. Let $p = \mu\nu$ denote the resulting joint distribution. Let $I_{\mu\nu} = I_{\mu\nu}(X;Y)$ be the average mutual information. Then $I_{\mu\nu}$ is a convex $\bigcup$ function of ν and a convex $\bigcap$ function of μ.*

Proof. The proof of convexity was suggested by T. Linder. Consider a fixed source μ and consider channels ν_1, ν_2, and $\nu = \lambda\nu_1 + (1-\lambda)\nu_2$. Denote the corresponding input/output pair processes by $p_i = \mu\nu_i$, $i = 1, 2$, and $p = \lambda p_1 + (1-\lambda)p_2$ and the corresponding output processes by η_i and $\eta = \lambda\eta_1 + (1-\lambda)\eta_2$, e.g., $\eta(G) = p(A_X^\infty \times G)$ for all output events G. Note that p_1, p_2, and p all have a common input distribution μ. We have that

$$\mu \times \eta = \lambda\mu \times \eta_1 + (1-\lambda)\mu \times \eta_2$$

so that from Lemma 7.2

$$\begin{aligned} I_{\mu\nu} &= D(\mu\nu||\mu \times \eta) = D(\lambda p_1 + (1-\lambda)p_2 \| \lambda\mu \times \eta_1 + (1-\lambda)\mu \times \eta_2) \\ &\le \lambda D(p_1\|\mu \times \eta_1) + (1-\lambda)D(p_2\|\mu \times \eta_2) \\ &= \lambda I_{\mu\nu_1} + (1-\lambda)I_{\mu\nu_2}, \end{aligned}$$

proving the convexity of mutual information with respect to the channel in direct imitation of the proof for the finite case.

The concavity with respect to the source distribution follows from the proof of the corresponding finite alphabet result, specifically the representation of (3.26), coupled with a sequence of asymptotically accurate quantizers for the input and output. As the quantization becomes asymptotically accurate, all of the terms in (3.26) converge upward to their limiting values, proving that (3.26) holds for the general distributions.

□

Next consider the mutual information $I(f(X), g(Y))$ for arbitrary measurable mappings f and g of X and Y. From Lemma 7.15 applied to the random variables $f(X)$ and $g(Y)$, this mutual information can be approximated arbitrarily closely by $I(q_1(f(X)); q_2(g(Y)))$ by an appropriate choice of quantizers q_1 and q_2. Since the composition of q_1 and f constitutes a finite quantization of X and similarly $q_2 g$ is a quantizer for Y, we must have that

$$I(f(X); g(Y)) \approx I(q_1(f(X)); q_2(g(Y)) \le I(X;Y).$$

Making this precise yields the following corollary.

Corollary 7.16. *If f is a measurable function of X and g is a measurable function of Y, then*

$$I(f(X), g(Y)) \le I(X;Y).$$

The corollary states that mutual information is reduced by any measurable mapping, whether finite or not. For practice we point out another proof of this basic result that directly applies a property of divergence. Let $P = P_{XY}$, $M = P_X \times P_Y$, and define the mapping $r(x,y) = (f(x), g(y))$. Then from Corollary 7.2 we have

$$I(X;Y) = D(P\|M) \ge D(P_r\|M_r) \ge D(P_{f(X),g(Y)}\|M_{f(X),g(Y)}).$$

But $M_{f(X),g(Y)} = P_{f(X)} \times P_{g(Y)}$ since

$$\begin{aligned} M_{f(X),g(Y)}(F_X \times F_Z) &= M(f^{-1}(F_X) \bigcap g^{-1}(F_Y) \\ &= P_X(f^{-1}(F_X)) \times P_Y(g^{-1}(F_Y)) \\ &= P_{f(X)}(F_X) \times P_{g(Y)}(F_Y). \end{aligned}$$

Thus the previous inequality yields the corollary. □

For the remainder of this section we focus on conditional entropy and information.

Although we cannot express mutual information as a difference of ordinary entropies in the general case (since the entropies of nondiscrete random variables are generally infinite), we can obtain such a

representation in the case where one of the two variables is discrete. Suppose we are given a joint distribution P_{XY} and that X is discrete. We can choose a version of the conditional probability given Y so that $p_{X|Y}(x|y) = P(X = x|Y = y)$ is a valid PMF (considered as a function of x for fixed y) with P_Y probability 1. (This follows from Corollary 5.8.1 of [55] since the alphabet of X is discrete; the alphabet of Y need not be even standard.) Define

$$H(X|Y = y) = \sum_x p_{X|Y}(x|y) \ln \frac{1}{p_{X|Y}(x|y)}$$

and

$$H(X|Y) = \int H(X|Y = y)\, dP_Y(y).$$

Note that this agrees with the formula of Section 3.6 in the case that both alphabets are finite. The following result is due to Wyner [197].

Lemma 7.20. *If X, Y are random variables and X has a finite alphabet, then*

$$I(X;Y) = H(X) - H(X|Y).$$

Proof. We first claim that $p_{X|Y}(x|y)/p_X(x)$ is a version of $dP_{XY}/d(P_X \times P_Y)$. To see this observe that for $F \in \mathcal{B}(A_X \times A_Y)$, letting F_y denote the section $\{x : (x, y) \in F\}$ we have that

$$\begin{aligned}\int_F \frac{p_{X|Y}(x|y)}{p_X(x)} d(P_X \times P_Y) &= \int \sum_{x \in F_y} \frac{p_{X|Y}(x|y)}{p_X(x)} p_X(x) dP_Y(y) \\ &= \int dP_Y(y) \sum_{x \in F_y} p_{X|Y}(x|y) \\ &= \int dP_Y(y) P_X(F_y|y) = P_{XY}(F).\end{aligned}$$

Thus

$$\begin{aligned}I(X;Y) &= \int \ln\left(\frac{p_{X|Y}(x|y)}{p_X(x)}\right) dP_{XY} \\ &= H(X) + \int dP_Y(y) \sum_x p_{X|Y}(x|y) \ln p_{X|Y}(x|y).\end{aligned}$$

□

We now wish to study the effects of quantizing on conditional information. As discussed in Section 3.6, it is not true that $I(X;Y|Z)$ is always greater than $I(f(X);q(Y)|r(Z))$ and hence that $I(X;Y|Z)$ can be written as a supremum over all quantizers and hence the definition of (7.30) and the formula (7.32) do not have the intuitive counterpart of a limit of informations of quantized values. We now consider an alternative (and

more general) definition of conditional mutual information due to Wyner [197]. The definition has the form of a supremum over quantizers and does not require the existence of the probability measure $P_{X\times Y|Z}$ and hence makes sense for alphabets that are not standard. Given P_{XYZ} and any finite measurements f and g on X and Y, we can choose a version of the conditional probability given $Z = z$ so that

$$p_z(a,b) = \Pr(f(X) = a, g(Y) = b|Z = z)$$

is a valid PMF with probability 1 (since the alphabets of f and g are finite and hence standard a regular conditional probability exists from Corollary 5.8.1 of [55] or Corollary 6.2 of [58]). For such finite measurements we can define

$$I(f(X);g(Y)|Z = z) = \sum_{a\in A_f}\sum_{b\in A_g} p_z(a,b)\ln\frac{p_z(a,b)}{\sum_{a'} p_z(a',b)\sum_{b'} p_z(a,b')},$$

that is, the ordinary discrete average mutual information with respect to the distribution p_z.

Lemma 7.21. *Define*

$$I'(X;Y|Z) = \sup_{f,g}\int dP_Z(z)I(f(X);g(Y)|Z = z),$$

where the supremum is over all quantizers. Then there exist sequences of quantizers (as in Lemma 7.18) such that

$$I'(X;Y|Z) = \lim_{n\to\infty} I'(q_m(X);r_m(Y)|Z).$$

I' satisfies Kolmogorov's formula, that is,

$$I'(X;Y|Z) = I((X,Z);Y) - I(Y;Z).$$

If the alphabets are standard, then

$$I(X;Y|Z) = I'(X;Y|Z).$$

Comment: The main point here is that conditional mutual information can be expressed as a supremum or limit of quantizers. The other results simply point out that the two conditional mutual informations have the same relation to ordinary mutual information and are (therefore) equal when both are defined. The proof follows Wyner [197].

Proof. First observe that for any quantizers q and r of A_f and A_g we have from the usual properties of mutual information that

$$I(q(f(X));r(g(Y))|Z = z) \le I(f(X);g(Y)|Z = z)$$

and hence integrating we have that

$$I'(q(f(X)); r(g(Y))|Z) = \int I(q(f(X)); r(g(Y))|Z = z)\, dP_Z(z)$$
$$\leq \int I(f(X); g(Y)|Z = z)\, dP_Z(z) \tag{7.42}$$

and hence taking the supremum over all q and r to get $I'(f(X); g(Y)|Z)$ yields

$$I'(f(X); g(Y)|Z) = \int I(f(X); g(Y)|Z = z)\, dP_Z(z). \tag{7.43}$$

so that (7.42) becomes

$$I'(q(f(X)); r(g(Y))|Z) \leq I'(f(X); g(Y)|Z) \tag{7.44}$$

for any quantizers q and r and the definition of I' can be expressed as

$$I'(X; Y|Z) = \sup_{f,g} I'(f(X); g(Y)|Z), \tag{7.45}$$

where the supremum is over all quantizers f and g. This proves the first part of the lemma since the supremum can be approached by a sequence of quantizers. Next observe that

$$I'(f(X); g(Y)|Z) = \int I(f(X); g(Y)|Z = z)\, dP_Z(z)$$
$$= H(g(Y)|Z) - H(g(Y)|f(X), Z).$$

Since we have from Lemma 7.20 that

$$I(g(Y); Z) = H(g(Y)) - H(g(Y)|Z),$$

we have by adding these equations and again using Lemma 7.20 that

$$I(g(Y); Z) + I'(f(X); g(Y)|Z) = H(g(Y)) - H(g(Y)|f(X), Z)$$
$$= I((f(X), Z); g(Y)).$$

Taking suprema over both sides over all quantizers f and g yields the relation

$$I(X; Z) + I'(X; Y|Z) = I((X, Z); Y),$$

proving Kolmogorov's formula. Lastly, if the spaces are standard, then from Kolmogorov's inequality for the original definition (which is valid for the standard space alphabets) combined with the above formula implies that

$$I'(X; Y|Z) = I((X, Z); Y) - I(X; Z) = I(X; Y|Z).$$

□

7.5 Convergence Results

We now combine the convergence results for divergence with the definitions and properties of information densities to obtain several convergence results for information densities. Unlike the results to come for relative entropy rate and information rate, these are results involving the information between a sequence of random variables and a fixed random variable.

Lemma 7.22. *Given random variables X and $Y_1, Y_2, \cdots$ defined on a common probability space,*

$$\lim_{n\to\infty} I(X;(Y_1,Y_2,\cdots,Y_n)) = I(X;(Y_1,Y_2,\cdots)).$$

If in addition $I(X;(Y_1,Y_2,\cdots)) < \infty$ and hence $P_X \times P_{Y_1,Y_2,\cdots} \gg P_{X,Y_1,Y_2,\cdots}$, then

$$i_{X;Y_1,Y_2,\cdots,Y_n} \underset{n\to\infty}{\to} i_{X;Y_1,Y_2,\cdots}$$

in Ł^1.

Proof. The first result follows from Corollary 7.5 with $X, Y_1, Y_2, \cdots, Y_{n-1}$ replacing X^n, P being the distribution $P_{X,Y_1,\cdots}$, and M being the product distribution $P_X \times P_{Y_1,Y_2,\cdots}$. The density result follows from Lemma 7.12. □

Corollary 7.17. *Given random variables X, Y, and $Z_1, Z_2, \cdots$ defined on a common probability space, then*

$$\lim_{n\to\infty} I(X;Y|Z_1,Z_2,\cdots,Z_n) = I(X;Y|Z_1,Z_2,\cdots).$$

If

$$I((X,Z_1,\cdots);Y) < \infty,$$

(e.g., if Y has a finite alphabet and hence $I((X,Z_1,\cdots);Y) \le H(Y) < \infty$), then also

$$i_{X;Y|Z_1,\cdots,Z_n} \underset{n\to\infty}{\to} i_{X;Y|Z_1,\cdots} \tag{7.46}$$

in L^1.

Proof. From Kolmogorov's formula

$$I(X;Y|Z_1,Z_2,\cdots,Z_n) = I(X;(Y,Z_1,Z_2,\cdots,Z_n)) - I(X;Z_1,\cdots,Z_n) \ge 0. \tag{7.47}$$

From the previous lemma, the first term on the left converges as $n \to \infty$ to $I(X;(Y,Z_1,\cdots))$ and the second term on the right is the negative of a term converging to $I(X;(Z_1,\cdots))$. If the first of these limits is finite, then the difference in (7.5) converges to the difference of these terms, which gives (7.46). From the chain rule for information densities, the conditional information density is the difference of the information densities:

$$i_{X;Y|Z_1,\cdots,Z_n} = i_{X;(Y,Z_1,\cdots,Z_n)} - i_{X;(Z_1,\cdots,Z_n)}$$

which is converging in L^1x to

$$i_{X;Y|Z_1,\cdots} = i_{X;(Y,Z_1,\cdots)} - i_{X;(Z_1,\cdots)},$$

again invoking the density chain rule. If $I(X;Y|Z_1,\cdots) = \infty$ then quantize Y as $q(Y)$ and note since $q(Y)$ has a finite alphabet that

$$I(X;Y|Z_1,Z_2,\cdots,Z_n) \ge I(X;q(Y)|Z_1,Z_2,\cdots,Z_n) \underset{n\to\infty}{\to} I(X;q(Y)|Z_1,\cdots)$$

and hence

$$\liminf_{N\to\infty} I(X;Y|Z_1,\cdots) \ge I(X;q(Y)|Z_1,\cdots).$$

Since the right-hand term above can be made arbitrarily large, the remaining part of the lemma is proved. □

Lemma 7.23. *If*

$$P_X \times P_{Y_1,Y_2,\cdots} \gg P_{X,Y_1,Y_2,\cdots}$$

(e.g., $I(X;(Y_1,Y_2,\cdots)) < \infty$), then with probability 1.

$$\lim_{n\to\infty} \frac{1}{n} i(X;(Y_1,\cdots,Y_n)) = 0.$$

Proof. This is a corollary of Theorem 7.4. Let P denote the distribution of $\{X,Y_1,Y_2,\cdots\}$ and let M denote the distribution $P_X \times P_{Y_1,\cdots}$. By assumption $M \gg P$. The information density is

$$i(X;(Y_1,\cdots,Y_n)) = \ln \frac{dP_n}{dM_n},$$

where P_n and M_n are the restrictions of P and M to $\sigma(X,Y_1,\cdots Y_n)$. Theorem 7.4 can therefore be applied to conclude that P-a.e.

$$\lim_{n\to\infty} \frac{1}{n} \ln \frac{dP_n}{dM_n} = 0,$$

which proves the lemma. □

The lemma has the following immediate corollary.

Corollary 7.18. *If $\{X_n\}$ is a process with the property that*

$$I(X_0; X_{-1}, X_{-2}, \cdots) < \infty,$$

that is, there is a finite amount of information between the zero time sample and the infinite past, then

$$\lim_{n\to\infty} \frac{1}{n} i(X_0; X_{-1}, \cdots, X_{-n}) = 0.$$

If the process is stationary, then also

$$\lim_{n\to\infty} \frac{1}{n} i(X_n; X^n) = 0.$$

Chapter 8
Information Rates

Abstract Definitions of information rate for processes with standard alphabets are developed and a mean ergodic theorem for information densities is proved. The relations among several different measures of information rate are developed.

8.1 Information Rates for Finite Alphabets

Let $\{(X_n, Y_n)\}$ be a one-sided random process with finite alphabet $A \times B$ and let $((A \times B)^{Z_+}, \mathcal{B}(A \times B)^{Z_+})$ be the corresponding one-sided sequence space of outputs of the pair process. We consider X_n and Y_n to be the sampling functions on the sequence spaces A^∞ and B^∞ and (X_n, Y_n) to be the pair sampling function on the product space, that is, for $(x, y) \in A^\infty \times B^\infty$, $(X_n, Y_n)(x, y) = (X_n(x), Y_n(y)) = (x_n, y_n)$. Let p denote the process distribution induced by the original space on the process $\{X_n, Y_n\}$. Analogous to entropy rate we can define the mutual information rate (or simply information rate) of a finite alphabet pair process by

$$\overline{I}(X, Y) = \limsup_{n \to \infty} \frac{1}{n} I(X^n, Y^n).$$

If the finite alphabet pair process is AMS, then

$$\overline{I}(X; Y) = \overline{H}(X) + \overline{H}(Y) - \overline{H}(X, Y) \tag{8.1}$$

and from Theorem 4.1 the entropy rates with respect to the AMS distribution equal those with respect to the stationary mean. These facts together with the properties of entropy rates of Theorems 3.3 and 4.1 yield the following lemma, where analogous to Theorem 4.1 we define the random variables $p(X^n, Y^n)$ by $p(X^n, Y^n)(x, y) = p(X^n = x^n, Y^n = y^n)$, $p(X^n)$ by $p(X^n)(x, y) = p(X^n = x^n)$, and similarly for $p(Y^n)$.

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Lemma 8.1. *Suppose that $\{X_n, Y_n\}$ is an AMS finite alphabet random process with distribution p and stationary mean $\overline{p}$. Then the limits supremum defining information rates are limits and*

$$\overline{I}_p(X,Y) = \overline{I}_{\overline{p}}(X,Y).$$

$\overline{I}_p$ is an affine function of the distribution p. If $\overline{p}$ has ergodic decomposition $\overline{p}_{xy}$, then

$$\overline{I}_p(X,Y) = \int d\overline{p}(x,y)\overline{I}_{\overline{p}_{xy}}(X,Y).$$

If we define the information density

$$i_n(X^n,Y^n) = \ln \frac{p(X^n,Y^n)}{p(X^n)p(Y^n)},$$

then

$$\lim_{n\to\infty} \frac{1}{n} i_n(X^n,Y^n) = \overline{I}_{\overline{p}_{xy}}(X,Y)$$

almost everywhere with respect to $\overline{p}$ and p and in $L^1(p)$.

The L^1 results are extensions of the results of Moy [127] and Perez [148] for stationary processes, which in turn extended the Shannon-McMillan theorem from entropies of discrete alphabet processes to information densities. See also Kieffer [97].

The following lemmas follow either directly from or similarly to the corresponding results for entropy rate of Section 6.1.

Lemma 8.2. *Suppose that $\{X_n, Y_n, X'_n, Y'_n\}$ is an AMS process and*

$$\overline{P} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \Pr((X_i,Y_i) \neq (X'_i,Y'_i)) \leq \epsilon$$

(the limit exists since the process is AMS). Then

$$|\overline{I}(X;Y) - \overline{I}(X';Y')| \leq 3(\epsilon \ln(\|A\| - 1) + h_2(\epsilon)).$$

Proof: The inequality follows from Corollary 6.1 since

$$\overline{|}(X;Y) - \overline{I}(X';Y')| \leq$$
$$|\overline{H}(X) - \overline{H}(X')| + |\overline{H}(Y) - \overline{H}(Y')| + |\overline{H}(X,Y) - \overline{H}(X',Y')|$$

and since $\Pr((X_i,Y_i) \neq (X_i',Y_i')) = \Pr(X_i \neq X_i' \text{ or } Y_i \neq Y_i')$ is no smaller than $\Pr(X_i \neq X_i')$ or $\Pr(Y_i \neq Y_i')$. □

Corollary 8.1. *Let $\{X_n, Y_n\}$ be an AMS process and let f and g be stationary measurements on X and Y, respectively. Given $\epsilon > 0$ there is an N sufficiently large, scalar quantizers q and r, and mappings f' and g' which*

depend only on $\{q(X_0), \cdots, q(X_{N-1})\}$ *and* $\{r(Y_0), \cdots, r(Y_{N-1})\}$ *in the one-sided case and* $\{q(X_{-N}), \cdots, q(X_N)\}$ *and* $\{r(Y_{-N}), \cdots, r(Y_N)\}$ *in the two-sided case such that*

$$|\bar{I}(f;g) - \bar{I}(f';g')| \leq \epsilon.$$

Proof: Choose the codes f' and g' from Lemma 5.2 and apply the previous lemma. □

Lemma 8.3. *If* $\{X_n, Y_n\}$ *is an AMS process and* f *and* g *are stationary codings of* X *and* Y*, respectively, then*

$$\bar{I}(X;Y) \geq \bar{I}(f;g).$$

Proof: This is proved as Corollary 6.4 by first approximating f and g by finite-window stationary codes, applying the result for mutual information (Lemma 3.12), and then taking the limit. □

8.2 Information Rates for General Alphabets

Suppose that we are given a pair random process $\{X_n, Y_n\}$ with distribution p. The most natural definition of the information rate between the two processes is the extension of the definition for the finite alphabet case:

$$\bar{I}(X;Y) = \limsup_{n\to\infty} \frac{1}{n} I(X^n; Y^n).$$

This was the first general definition of information rate and it is due to Dobrushin [32]. While this definition has its uses, it also has its problems. Another definition is more in the spirit of the definition of information itself: We formed the general definitions by taking a supremum of the finite alphabet definitions over all finite-alphabet codings or quantizers. The above definition takes the limit of such suprema. An alternative definition is to instead reverse the order and take the supremum of the limit and hence the supremum of the information rate over all finite-alphabet codings of the process. This reversal of supremum and limit provides a definition of information rate similar to the definition of the entropy of a dynamical system. There is a question as to what kind of codings we permit, that is, do the quantizers quantize individual outputs or long sequences of outputs. We shall shortly see that it makes no difference. Suppose that we have a pair random process $\{X_n, Y_n\}$ with standard alphabets A_X and A_Y and suppose that $f : A_X^\infty \to A_f$ and $g : A_Y^\infty \to A_g$ are stationary codings of the X and Y sequence spaces into a finite alphabet. Let $\{f_n, g_n\}$ be the induced output process, that is, if T denotes the shift (on any of the sequence spaces) then $f_n(x, y) = f(T^n x)$ and

$g_n(x,y) = g(T^n y)$. Recall that $f(T^n(x,y)) = f_n(x,y)$, that is, shifting the input n times results in the output being shifted n times.

Since the new process $\{f_n, g_n\}$ has a finite alphabet, its mutual information rate is defined. We define the information rate for general alphabets by

$$\begin{aligned} I^*(X;Y) &= \sup_{\text{sliding-block codes } f,g} \overline{I}(f;g) \\ &= \sup_{\text{sliding-block codes } f,g} \limsup_{n\to\infty} \frac{1}{n} I(f^n;g^n). \end{aligned}$$

We now focus on AMS processes, in which case the information rates for finite-alphabet processes (e.g., quantized processes) is given by the limit, that is,

$$\begin{aligned} I^*(X;Y) &= \sup_{\text{sliding-block codes } f,g} \overline{I}(f;g) \\ &= \sup_{\text{sliding-block codes } f,g} \lim_{n\to\infty} \frac{1}{n} I(f^n;g^n). \end{aligned}$$

Note for later use the following simple inequality which follows from the the above facts and Corollary 7.11. As with that corollary, the result is useful if the entropy rate of one component of the process is finite.

Lemma 8.4. *Given an AMS pair process (X,Y) with standard alphabets,*

$$I^*(X;Y) \le \overline{H}(Y).$$

The following lemma shows that for AMS sources I^* can also be evaluated by constraining the sliding-block codes to be scalar quantizers.

Lemma 8.5. *Given an AMS pair random process $\{X_n, Y_n\}$ with standard alphabet,*

$$I^*(X;Y) = \sup_{q,r} \overline{I}(q(X);r(Y)) = \sup_{q,r} \limsup_{n\to\infty} \frac{1}{n} I(q(X)^n; r(Y)^n),$$

where the supremum is over all quantizers q of A_X and r of A_Y and where $q(X)^n = q(X_0), \cdots, q(X_{n-1})$.

Proof: Clearly the right hand side above is less than I^* since a scalar quantizer is a special case of a stationary code. Conversely, suppose that f and g are sliding-block codes such that $\overline{I}(f;g) \ge I^*(X;Y) - \epsilon$. Then from Corollary 8.1 there are quantizers q and r and codes f' and g' depending only on the quantized processes $q(X_n)$ and $r(Y_n)$ such that $\overline{I}(f';g') \ge \overline{I}(f;g) - \epsilon$. From Lemma 8.3, however, $\overline{I}(q(X);r(Y)) \ge \overline{I}(f';g')$ since f' and g' are stationary codings of the quantized pro-

cesses. Thus $\bar{I}(q(X);r(Y)) \geq I^*(X;Y) - 2\epsilon$, which proves the lemma. □

Corollary 8.2.

$$I^*(X;Y) \leq \bar{I}(X;Y).$$

If the alphabets are finite, then the two rates are equal.

Proof: The inequality follows from the lemma and the fact that

$$I(X^n;Y^n) \geq I(q(X)^n;r(Y)^n)$$

for any scalar quantizers q and r (where $q(X)^n$ is $q(X_0), \cdots, q(X_{n-1})$). If the alphabets are finite, then the identity mappings are quantizers and yield $I(X^n;Y^n)$ for all n. □

Pinsker [150] introduced the definition of information rate as a supremum over all scalar quantizers and hence we refer to this information rate as the Pinsker rate. The Pinsker definition has the advantage that we can use the known properties of information rates for finite-alphabet processes to infer those for general processes, an attribute the first definition lacks.

Corollary 8.3. *Given a standard alphabet pair process alphabet $A_X \times A_Y$, there is a sequence of scalar quantizers q_m and r_m such that for any AMS pair process $\{X_n, Y_n\}$ having this alphabet (that is, for any process distribution on the corresponding sequence space)*

$$I(X^n;Y^n) = \lim_{m\to\infty} I(q_m(X)^n;r_m(Y)^n)$$

$$I^*(X;Y) = \lim_{m\to\infty} \bar{I}(q_m(X);r_m(Y)).$$

Furthermore, the above limits can be taken to be increasing by using finer and finer quantizers.

Comment: It is important to note that the same sequence of quantizers gives both of the limiting results.
Proof: The first result is Lemma 7.18. The second follows from the previous lemma. □

Observe that

$$I^*(X;Y) = \lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{n} I(q_m(X);r_m(Y))$$

whereas

$$\bar{I}(X;Y) = \limsup_{n\to\infty} \lim_{m\to\infty} \frac{1}{n} I(q_m(X);r_m(Y)).$$

Thus the two notions of information rate are equal if the two limits can be interchanged. We shall later consider conditions under which this is

true and we shall see that equality of these two rates is important for proving ergodic theorems for information densities.

Lemma 8.6. *Suppose that $\{X_n, Y_n\}$ is an AMS standard alphabet random process with distribution p and stationary mean $\overline{p}$. Then*

$$I_p^*(X;Y) = I_{\overline{p}}^*(X;Y).$$

I_p^* is an affine function of the distribution p. If $\overline{p}$ has ergodic decomposition $\overline{p}_{xy}$, then

$$I_p^*(X;Y) = \int d\overline{p}(x,y) I^*{}_{\overline{p}_{xy}}(X;Y).$$

If f and g are stationary codings of X and Y, then

$$I_p^*(f;g) = \int d\overline{p}(x,y) I_{\overline{p}_{xy}}^*(f;g).$$

Proof: For any scalar quantizers q and r of X and Y we have that $\overline{I}_p(q(X);r(Y)) = \overline{I}_{\overline{p}}(q(X);r(Y))$. Taking a limit with ever finer quantizers yields the first equality. The fact that I^* is affine follows similarly. Suppose that $\overline{p}$ has ergodic decomposition $\overline{p}_{xy}$. Define the induced distributions of the quantized process by m and m_{xy}, that is, $m(F) = \overline{p}(x,y : \{q(x_i), r(y_i);\, i \in \mathbb{T}\} \in F)$ and similarly for m_{xy}. The m_{xy} are stationary and ergodic since they are stationary codings of stationary ergodic processes and together they form an ergodic decomposition of m, which must also be stationary. Let X'_n, Y'_n denote the coordinate functions on the quantized output sequence space (that is, the processes $\{q(X_n), r(Y_n)\}$ and $\{X'_n, Y'_n\}$ are equivalent), then using the ergodic decomposition of mutual information for finite-alphabet processes (Lemma 8.1) we have that

$$\begin{aligned}\overline{I}_p(q(X);r(Y)) &= \overline{I}_m(X';Y') = \int \overline{I}_{m_{x'y'}}(X';Y')\, dm(x',y') \\ &= \int \overline{I}_{\overline{p}_{xy}}(q(X);r(Y))\, d\overline{p}(x,y).\end{aligned}$$

Replacing the quantizers by the sequence q_m, r_m the result then follows by taking the limit using the monotone convergence theorem. The result for stationary codings follows similarly by applying the previous result to the induced distributions and then relating the equation to the original distributions. □

The above properties are not known to hold for $\overline{I}$ in the general case. Thus although $\overline{I}$ may appear to be a more natural definition of mutual information rate, I^* is better behaved since it inherits properties from the discrete alphabet case. It will be of interest to find conditions under which the two rates are the same, since then $\overline{I}$ will share the properties

possessed by I^*. The first result of the next section adds to the interest by demonstrating that when the two rates are equal, a mean ergodic theorem holds for the information densities.

8.3 A Mean Ergodic Theorem for Densities

Theorem 8.1. *Given an AMS pair process $\{X_n, Y_n\}$ with standard alphabets, assume that for all n*

$$P_{X^n} \times P_{Y^n} \gg P_{X^nY^n}$$

and hence that the information densities

$$i_{X^n;Y^n} = \ln \frac{dP_{X^n,Y^n}}{d(P_{X^n} \times P_{Y^n})}$$

are well defined. For simplicity we abbreviate $i_{X^n;Y^n}$ to i_n when there is no possibility of confusion. If the limit

$$\lim_{n\to\infty} \frac{1}{n} I(X^n;Y^n) = \overline{I}(X;Y)$$

exists and

$$\overline{I}(X;Y) = I^*(X;Y) < \infty,$$

then $n^{-1}i_n(X^n;Y^n)$ converges in L^1 to an invariant function $i(X;Y)$. If the stationary mean of the process has an ergodic decomposition $\overline{p}_{xy}$, then the limiting density is $I^*_{\overline{p}_{xy}}(X;Y)$, the information rate of the ergodic component in effect.

Proof: Let q_m and r_m be asymptotically accurate quantizers for A_X and A_Y. Define the discrete approximations $\hat{X}_n = q_m(X_n)$ and $\hat{Y}_n = r_m(Y_n)$. Observe that $P_{X^n} \times P_{Y^n} \gg P_{X^nY^n}$ implies that $P_{\hat{X}^n} \times P_{\hat{Y}^n} \gg P_{\hat{X}^n\hat{Y}^n}$ and hence we can define the information densities of the quantized vectors by

$$\hat{i}_n = \ln \frac{dP_{\hat{X}_n\hat{Y}^n}}{d(P_{\hat{X}^n} \times P_{\hat{Y}^n})}.$$

For any m we have that

$$\int |\frac{1}{n} i_n(x^n;y^n) - I^*{}_{\overline{p}_{xy}}(X;Y)|\, dp(x,y) \le$$
$$\int |\frac{1}{n} i_n(x^n;y^n) - \frac{1}{n}\hat{i}_n(q_m(x)^n;r_m(y)^n)|\, dp(x,y) +$$
$$\int |\frac{1}{n}\hat{i}_n(q_m(x)^n;r_m(y)^n) - \overline{I}_{\overline{p}_{xy}}(q_m(X);r_m(Y))|\, dp(x,y) +$$
$$\int |\overline{I}_{\overline{p}_{xy}}(q_m(X);r_m(Y)) - I^*{}_{\overline{p}_{xy}}(X;Y)|\, dp(x,y), \quad (8.2)$$

where

$$q_m(x)^n = (q_m(x_0), \cdots, q_m(x_{n-1})),$$
$$r_m(y)^n = (r_m(y_0), \cdots, r_m(y_{n-1})),$$

and $\overline{I}_p(q_m(X);r_m(Y))$ denotes the information rate of the process

$$\{q_m(X_n), r_m(Y_n); n = 0, 1, \cdots,\}$$

when p is the process measure describing $\{X_n, Y_n\}$.

Consider first the right-most term of (8.2). Since I^* is the supremum over all quantized versions,

$$\int |\overline{I}_{\overline{p}_{xy}}(q_m(X);r_m(Y)) - I^*_{\overline{p}_{xy}}(X;Y)|\, dp(x,y) =$$
$$\int (I^*_{\overline{p}_{xy}}(X;Y) - \overline{I}_{\overline{p}_{xy}}(q_m(X);r_m(Y)))\, dp(x,y).$$

Using the ergodic decomposition of I^* (Lemma 8.6) and that of $\overline{I}$ for discrete alphabet processes (Lemma 8.1) this becomes

$$\int |\overline{I}_{\overline{p}_{xy}}(q_m(X);r_m(Y)) - I^*_{\overline{p}_{xy}}(X;Y)|\, dp(x,y) =$$
$$I^*_p(X;Y) - \overline{I}_p(q_m(X);r_m(Y)). \quad (8.3)$$

For fixed m the middle term of (8.2) can be made arbitrarily small by taking n large enough from the finite alphabet result of Lemma 8.1. The first term on the right can be bounded above using Corollary 7.6 with $\mathcal{F} = \sigma(q(X)^n; r(Y)^n)$ by

$$\frac{1}{n}\left(I(X^n;Y^n) - I(\hat{X}^n;\hat{Y}^n) + \frac{2}{e}\right)$$

which as $n \to \infty$ goes to $\overline{I}(X;Y) - \overline{I}(q_m(X);r_m(Y))$. Thus we have for any m that

$$\limsup_{n\to\infty} \int |\frac{1}{n} i_n(x^n;y^n) - I^*_{\overline{p}_{xy}}(X;Y)|\, dp(x,y) \le$$
$$\overline{I}(X;Y) - \overline{I}(q_m(X);r_m(Y)) + I^*(X;Y) - \overline{I}(q_m(X);r_m(Y))$$

which as $m \to \infty$ becomes $\overline{I}(X;Y) - I^*(X;Y)$, which is 0 by assumption. □

8.4 Information Rates of Stationary Processes

In this section we introduce two more definitions of information rates for the case of stationary two-sided processes. These rates are useful tools in relating the Dobrushin and Pinsker rates and they provide additional interpretations of mutual information rates in terms of ordinary mutual information. The definitions follow Pinsker [150].

Henceforth assume that $\{X_n, Y_n\}$ is a stationary two-sided pair process with standard alphabets. Define the sequences $\mathcal{Y} = \{\mathcal{Y}_i; i \in \mathbb{T}\}$ and $Y = \{Y_i; i \in \mathbb{T}\}$

First define

$$\tilde{I}(X;Y) = \limsup_{n\to\infty} \frac{1}{n} I(X^n;Y),$$

that is, consider the per-letter limiting information between n-tuples of X and the entire sequence from Y. Next define

$$I^-(X;Y) = I(X_0;Y|X_{-1},X_{-2},\cdots),$$

that is, the average conditional mutual information between one letter from X and the entire Y sequence given the infinite past of the X process. We could define the first rate for one-sided processes, but the second makes sense only when we can consider an infinite past. For brevity we write $X^- = X_{-1}, X_{-2}, \cdots$ and hence

$$I^-(X;Y) = I(X_0;Y|X^-).$$

Theorem 8.2.

$$\tilde{I}(X;Y) \ge \overline{I}(X;Y) \ge I^*(X;Y) \ge I^-(X;Y).$$

If the alphabet of X is finite, then the above rates are all equal.

Comment: We will later see more general sufficient conditions for the equality of the various rates, but the case where one alphabet is finite is simple and important and points out that the rates are all equal in the finite alphabet case.

Proof: We have already proved the middle inequality. The left inequality follows immediately from the fact that $I(X^n;Y) \geq I(X^n;Y^n)$ for all n. The remaining inequality is more involved. We prove it in two steps. First we prove the second half of the theorem, that the rates are the same if X has finite alphabet. We then couple this with an approximation argument to prove the remaining inequality. Suppose now that the alphabet of X is finite. Using the chain rule and stationarity we have that

$$\begin{aligned}\frac{1}{n}I(X^n;Y^n) &= \frac{1}{n}\sum_{i=0}^{n-1} I(X_i;Y^n|X_0,\cdots,X_{i-1}) \\ &= \frac{1}{n}\sum_{i=0}^{n-1} I(X_0;Y_{-i}^n|X_{-1},\cdots,X_{-i}),\end{aligned}$$

where Y_{-i}^n is $Y_{-i},\cdots,Y_{-i+n-1}$, that is, the n-vector starting at $-i$. Since X has finite alphabet, each term in the sum is bounded. We can show as in Section 7.4 (or using Kolmogorov's formula and Lemma 7.14) that each term converges as $i \to \infty$, $n \to \infty$, and $n-i \to \infty$ to $I(X_0;Y|X_{-1},X_{-2},\cdots)$ or $I^-(X;Y)$. These facts, however, imply that the above Cesàro average converges to the same limit and hence $\overline{I} = I^-$. We can similarly expand $\tilde{I}$ as

$$\frac{1}{n}\sum_{i=0}^{n-1} I(X_i;Y|X_0,\cdots,X_{i-1}) = \frac{1}{n}\sum_{i=0}^{n-1} I(X_0;Y|X_{-1},\cdots,X_{-i}),$$

which converges to the same limit for the same reasons. Thus $\tilde{I} = \overline{I} = I^-$ for stationary processes when the alphabet of X is finite. Now suppose that X has a standard alphabet and let q_m be an asymptotically accurate sequences of quantizers. Recall that the corresponding partitions are increasing, that is, each refines the previous partition. Fix $\epsilon > 0$ and choose m large enough so that the quantizer $\alpha(X_0) = q_m(X_0)$ satisfies

$$I(\alpha(X_0);Y|X^-) \geq I(X_0;Y|X^-) - \epsilon.$$

Observe that so far we have only quantized X_0 and not the past. Since

$$\mathcal{F}_m = \sigma(\alpha(X_0), Y, q_m(X_{-i});\ i = 1,2,\cdots)$$

asymptotically generates

$$\sigma(\alpha(X_0), Y, X_{-i};\ i = 1,2,\cdots),$$

given ϵ we can choose for m large enough (larger than before) a quantizer $\beta(x) = q_m(x)$ such that if we define $\beta(X^-)$ to be $\beta(X_{-1}), \beta(X_{-2}), \cdots$, then

$$|I(\alpha(X_0);(Y,\beta(X^-))) - I(\alpha(X_0);(Y,X^-))| \leq \epsilon$$

and

$$|I(\alpha(X_0);\beta(X^-)) - I(\alpha(X_0);X^-)| \le \epsilon.$$

Using Kolmogorov's formula this implies that

$$|I(\alpha(X_0);Y|X^-) - I(\alpha(X_0);Y|\beta(X^-))| \le 2\epsilon$$

and hence that

$$I(\alpha(X_0);Y|\beta(X^-)) \ge I(\alpha(X_0);Y|X^-) - 2\epsilon \ge I(X_0;Y|X^-) - 3\epsilon.$$

But the partition corresponding to β refines that of α and hence increases the information; hence

$$I(\beta(X_0);Y|\beta(X^-)) \ge I(\alpha(X_0);Y|\beta(X^-)) \ge I(X_0;Y|X^-) - 3\epsilon.$$

Since $\beta(X_n)$ has a finite alphabet, however, from the finite alphabet result the left-most term above must be $\bar{I}(\beta(X);Y)$, which can be made arbitrarily close to $I^*(X;Y)$. Since ϵ is arbitrary, this proves the final inequality. □

The following two theorems provide sufficient conditions for equality of the various information rates. The first result is almost a special case of the second, but it is handled separately as it is simpler, much of the proof applies to the second case, and it is not an exact special case of the subsequent result since it does not require the second condition of that result. The result corresponds to condition (7.4.33) of Pinsker [150], who also provides more general conditions. The more general condition is also due to Pinsker and strongly resembles that considered by Barron [8].

Theorem 8.3. *Given a stationary pair process $\{X_n, Y_n\}$ with standard alphabets, if*

$$I(X_0;(X_{-1},X_{-2},\cdots)) < \infty,$$

then

$$\tilde{I}(X;Y) = \bar{I}(X;Y) = I^*(X;Y) = I^-(X;Y). \tag{8.4}$$

Proof: We have that

$$\frac{1}{n}I(X^n;Y) \le \frac{1}{n}I(X^n;(Y,X^-)) = \frac{1}{n}I(X^n;X^-) + \frac{1}{n}I(X^n;Y|X^-), \tag{8.5}$$

where, as before, $X^- = \{X_{-1},X_{-2},\cdots\}$. Consider the first term on the right. Using the chain rule for mutual information

$$\frac{1}{n}I(X^n;X^-) = \frac{1}{n}\sum_{i=0}^{n-1} I(X_i;X^-|X^i) \tag{8.6}$$

$$= \frac{1}{n}\sum_{i=0}^{n-1} (I(X_i;(X^i,X^-)) - I(X_i;X^i)). \tag{8.7}$$

Using stationarity we have that

$$\frac{1}{n}I(X^n;X^-) = \frac{1}{n}\sum_{i=0}^{n-1} (I(X_0;X^-) - I(X_0;(X_{-1},\cdots,X_{-i})). \tag{8.8}$$

The terms $I(X_0;(X_{-1},\cdots,X_{-i}))$ are converging to $I(X_0;X^-)$, hence the terms in the sum are converging to 0, i.e.,

$$\lim_{i\to\infty} I(X_i;X^-|X^i) = 0. \tag{8.9}$$

The Cesàro mean of (8.7) is converging to the same thing and hence

$$\frac{1}{n}I(X^n;X^-) \to 0. \tag{8.10}$$

Next consider the term $I(X^n;Y|X^-)$. For any positive integers n,m we have

$$I(X^{n+m};Y|X^-) = I(X^n;Y|X^-) + I(X_n^m;Y|X^-,X^n), \tag{8.11}$$

where $X_n^m = X_n,\cdots,X_{n+m-1}$. From stationarity, however, the rightmost term is just $I(X^m;Y|X^-)$ and hence

$$I(X^{m+n};Y|X^-) = I(X^n;Y|X^-) + I(X^m;Y|X^-). \tag{8.12}$$

This is just a linear functional equation of the form $f(n+m) = f(n) + f(m)$ and the unique solution to such an equation is $f(n) = nf(1)$, that is,

$$\frac{1}{n}I(X^n;Y|X^-) = I(X_0;Y|X^-) = I^-(X;Y). \tag{8.13}$$

Taking the limit supremum in (8.5) yields

$$\tilde{I}(X;Y) \le I^-(X;Y), \tag{8.14}$$

which with Theorem 8.2 completes the proof. □

Intuitively, the theorem states that if one of the processes has finite average mutual information between one symbol and its infinite past, then the Dobrushin and Pinsker information rates yield the same value and hence there is an L^1 ergodic theorem for the information density.

To generalize the theorem we introduce a condition that will often be useful when studying asymptotic properties of entropy and informa-

tion. A stationary process $\{X_n\}$ is said to have the *finite-gap information property* if there exists an integer K such that

$$I(X_K; X^- | X^K) < \infty, \tag{8.15}$$

where, as usual, $X^- = (X_{-1}, X_{-2}, \cdots)$. When a process has this property for a specific K, we shall say that it has the K-gap information property. Observe that if a process possesses this property, then it follows from Lemma 7.17

$$I(X_K; (X_{-1}, \cdots, X_{-l}) | X^K) < \infty; \; l = 1, 2, \cdots \tag{8.16}$$

Since these informations are finite,

$$P_{X^n}^{(K)} \gg P_{X^n}; \; n = 1, 2, \ldots, \tag{8.17}$$

where $P_{X^n}^{(K)}$ is the Kth order Markov approximation to P_{X^n}.

Theorem 8.4. *Given a stationary standard alphabet pair process $\{X_n, Y_n\}$, if $\{X_n\}$ satisfies the finite-gap information property (8.15) and if, in addition,*

$$I(X^K; Y) < \infty, \tag{8.18}$$

then (8.4) holds.

If $K = 0$ then there is no conditioning and (8.18) is trivial, that is, the previous theorem is the special case with $K = 0$.

Comment: This theorem shows that if there is any finite dimensional future vector $(X_K, X_{K+1}, \cdots, X_{K+N-1})$ which has finite mutual information with respect to the infinite past X^- when conditioned on the intervening gap $(X_0, \cdots, X_{K-1})$, then the various definitions of mutual information are equivalent provided that the mutual information betwen the "gap" X^K and the sequence $\mathbf{Y}$ are finite. Note that this latter condition will hold if, for example, $\tilde{I}(X; Y)$ is finite.

Proof: For $n > K$

$$\frac{1}{n} I(X^n; Y) = \frac{1}{n} I(X^K; Y) + \frac{1}{n} I(X_K^{n-K}; Y | X^K).$$

By assumption the first term on the left will tend to 0 as $n \to \infty$ and hence we focus on the second, which can be broken up analogous to the previous theorem with the addition of the conditioning:

$$\begin{aligned} \frac{1}{n} I(X_K^{n-K}; Y | X^K) &\le \frac{1}{n} I(X_K^{n-K}; (Y, X^- | X^K)) \\ &= \frac{1}{n} I(X_K^{n-K}; X^- | X^K) + \frac{1}{n} I(X_K^{n-K}; Y | X^-, X^K). \end{aligned}$$

Consider first the term

$$\frac{1}{n} I(X_K^{n-K}; X^- | X^K) = \frac{1}{n} \sum_{i=K}^{n-1} I(X_i; X^- | X^i),$$

which is as (8.7) in the proof of Theorem 8.3 except that the first K terms are missing. The same argument then shows that the limit of the sum is 0. The remaining term is

$$\frac{1}{n} I(X_K^{n-K}; Y | X^-, X^K) = \frac{1}{n} I(X^n; Y | X^-)$$

exactly as in the proof of Theorem 8.3 and the same argument then shows that the limit is $I^-(X;Y)$, which completes the proof. □

One result developed in the proofs of Theorems 8.3 and 8.4 will be important later in its own right and hence we isolate it as a corollary. The result is just (8.9), which remains valid under the more general conditions of Theorem 8.4, and the fact that the Cesàro mean of converging terms has the same limit.

Corollary 8.4. *If a process $\{X_n\}$ has the finite-gap information property*

$$I(X_K; X^- | X^K) < \infty$$

for some K, then

$$\lim_{n\to\infty} I(X_n; X^- | X^n) = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n} I(X^n; X^-) = 0.$$

The corollary can be interpreted as saying that if a process has the the finite gap information property, then the mutual information between a single sample and the infinite past conditioned on the intervening samples goes to zero as the number of intervening samples goes to infinity. This can be interpreted as a form of asymptotic independence property of the process.

Corollary 8.5. *If a one-sided stationary source $\{X_n\}$ is such that for some K, $I(X_n; X^{n-K} | X_{n-K}^K)$ is bounded uniformly in n, then it has the finite-gap property and hence*

$$\bar{I}(X;Y) = I^*(X;Y).$$

Proof: Simply imbed the one-sided source into a two-sided stationary source with the same probabilities on all finite-dimensional events. For that source

$$I(X_n; X^{n-K} | X_{n-K}^K) = I(X_K; X_{-1}, \cdots, X_{-n-K} | X^K) \underset{n\to\infty}{\to} I(X_K; X^- | X^K).$$

Thus if the terms are bounded, the conditions of Theorem 8.3 are met for the two-sided source. The one-sided equality then follows. □

The above results have an information theoretic implication for the ergodic decomposition, which is described in the next theorem.

Theorem 8.5. *Suppose that $\{X_n\}$ is a stationary process with the finite-gap property (8.15). Let ψ be the ergodic component function of Theorem 1.6 and suppose that for some n*

$$I(X^n;\psi) < \infty. \tag{8.19}$$

(This will be the case, for example, if the finite-gap information property holds for 0 gap, that is, $I(X_0;X^-) < \infty$ since ψ can be determined from X^- and information is decreased by taking a function.) Then

$$\lim_{n\to\infty} \frac{1}{n} I(X^n;\psi) = 0.$$

Comment: For discrete alphabet processes this theorem is just the ergodic decomposition of entropy rate in disguise (Theorem 3.3). It also follows for finite-alphabet processes from Lemma 4.3. We shall later prove a corresponding almost everywhere convergence result for the corresponding densities. All of these results have the interpretation that the per-symbol mutual information between the outputs of the process and the ergodic component decreases with time because the ergodic component in effect can be inferred from the process output in the limit of an infinite observation sequence. The finiteness condition on some $I(X^n;\psi)$ is necessary for the nonzero finite-gap case to avoid cases such as where $X_n = \psi$ for all n and hence

$$I(X^n;\psi) = I(\psi;\psi) = H(\psi) = \infty,$$

in which case the theorem does not hold.

Proof:

Define $\psi_n = \psi$ for all n. Since ψ is invariant, $\{X_n, \psi_n\}$ is a stationary process. Since X_n satisfies the given conditions, however, $\bar{I}(X;\psi) = I^*(X;\psi)$. But for any scalar quantizer q, $\bar{I}(q(X);\psi)$ is 0 from Lemma 4.3. $I^*(X;\psi)$ is therefore 0 since it is the supremum of $\bar{I}(q(X);\psi)$ over all quantizers q. Thus

$$0 = \bar{I}(X;\psi) = \lim_{n\to\infty} \frac{1}{n} I(X^n;\psi^n) = \lim_{n\to\infty} \frac{1}{n} I(X^n;\psi). \quad \square$$

8.5 The Data Processing Theorem

The following is a basic property of a communication system. *If a communication system is stationary, then the mutual information rate between the overall input and output cannot exceed that over the channel.* The result is often called the *data processing theorem*.

Lemma 8.7. *Suppose that a communication system is stationary in the sense that the process* $\{X_n, U_n, Y_n, \hat{X}_n\}$ *is stationary. Then*

$$\tilde{I}(U;Y) \geq \bar{I}(X;Y) \geq \bar{I}(X;\hat{X}). \tag{8.20}$$

If $\{U_n\}$ *has a finite alphabet or if it has has the* K*-gap information property (8.15) and* $I(U^K, Y) < \infty$*, then*

$$\bar{I}(X;\hat{X}) \leq \bar{I}(U;Y).$$

Proof: Since $\{\hat{X}_n\}$ is a stationary deterministic encoding of the $\{Y_n\}$ $\bar{I}(X;\hat{X}) \leq I^*(X;Y)$. From Theorem 8.2 the right hand side is bounded above by $\bar{I}(X;Y)$. For each n

$$\begin{aligned} I(X^n;Y^n) &\leq I((X^n,U);Y^n) \\ &= I(Y^n;U) + I(X^n;Y^n|U) = I(Y^n;U), \end{aligned}$$

where $U = \{U_n, n \in \mathbb{T}\}$ and we have used the fact that $X \to U \to Y$ is a Markov chain and hence so is $X^N \to U \to Y^K$ and hence the conditional mutual information is 0 (Lemma 7.15). Thus

$$\bar{I}(X;Y) \leq \lim_{n\to\infty} I(Y^n;U) = \tilde{I}(Y;U).$$

Applying Theorem 8.2 then proves that

$$\bar{I}(X;\hat{X}) \leq \tilde{I}(Y;U).$$

If $\{U_n\}$ has finite alphabet or has the K-gap information property and $I(U^K, Y) < \infty$, then from Theorems 8.2 or 8.4, respectively, $\tilde{I}(Y;U) = \bar{I}((Y;U)$, completing the proof. □

The lemma can be easily extended to block stationary processes.

Corollary 8.6. *Suppose that the process of the previous lemma is not stationary, but is* (N,K)*-stationary in the sense that the vector process* $\{X_{nN}^N, U_{nK}^K, Y_{nK}^K, \hat{X}_{nN}^N\}$ *is stationary. Then*

$$\bar{I}(X;\hat{X}) \leq \frac{K}{N}\bar{I}(U;Y).$$

Proof: Apply the previous lemma to the stationary vector sequence to conclude that

$$\bar{I}(X^N; \hat{X}^N) \le \bar{I}(U^K; Y^K).$$

But

$$\bar{I}(X^N; \hat{X}^N) = \lim_{n\to\infty} \frac{1}{n} I(X^{nN}; \hat{X}^{nN}) = \lim_{n\to\infty} E\left[n^{-1} i_{X^{nN},\hat{X}^{nN}}\right],$$

which is N times the limiting expectation of a subsequence of the densities $n^{-1} i_{X^n,\hat{X}^n}$, whose expectation converges to $\bar{I}(X;Y)$. Thus

$$\bar{I}(X^N; X^N) = N\bar{I}(X; \hat{X}).$$

A similar manipulation for $\bar{I}(U^K; Y^K)$ completes the proof. □

8.6 Memoryless Channels and Sources

A useful inequality is developed in this section for the mutual information between the input and output of a memoryless channel. For contrast we also describe the corresponding result for a memoryless source and an arbitrary channel.

Lemma 8.8. *Let $\{X_n\}$ be a source with distribution μ and let ν be a channel. Let $\{X_n, Y_n\}$ be the hookup with distribution p. If the channel is memoryless, then for any n*

$$I(X^n; Y^n) \le \sum_{i=0}^{n-1} I(X_i; Y_i)$$

If instead the source is memoryless, then the inequality is reversed:

$$I(X^n; Y^n) \ge \sum_{i=0}^{n-1} I(X_i; Y_i).$$

Thus if both source and channel are memoryless,

$$I(X^n; Y^n) = \sum_{i=0}^{n-1} I(X_i; Y_i).$$

Proof: First suppose that the process is discrete. Then

$$I(X^n; Y^n) = H(Y^n) - H(Y^n|X^n).$$

Since by construction

$$P_{Y^n|X^n}(y^n|x^n) = \prod_{i=0}^{n-1} P_{Y_0|X_0}(y_i|x_i)$$

an easy computation shows that

$$H(Y^n|X^n) = \sum_{i=0}^{n-1} H(Y_i|X_i).$$

This combined with the inequality

$$H(Y^n) \le \sum_{i=0}^{n-1} H(Y_i)$$

(Lemma 3.2 used several times) completes the proof of the memoryless channel result for finite alphabets. If instead the source is memoryless, we have

$$I(X^n;Y^n) = H(X^n) - H(X^n|Y^n) = \sum_{i=0}^{n-1} H(X_i) - H(X^n|Y^n).$$

Extending Lemma 3.2 to conditional entropy yields

$$H(X^n|Y^n) \le \sum_{i=0}^{n-1} H(X_i|Y^n)$$

which can be further overbounded by using Lemma 3.12 (the fact that reducing conditioning increases conditional entropy) as

$$H(X^n|Y^n) \le \sum_{i=0}^{n-1} H(X_i|Y_i)$$

which implies that

$$I(X^n;Y^n) \ge \sum_{i=0}^{n-1} H(X_i) - H(X_i|Y_i) = \sum_{i=0}^{n-1} I(X_i;Y_i),$$

which completes the proof for finite alphabets.

To extend the result to standard alphabets, first consider the case where the Y^n are quantized to a finite alphabet. If the Y_k are conditionally independent given X^k, then the same is true for $q(Y_k)$, $k = 0, 1, \cdots, n-1$. Lemma 7.20 then implies that as in the discrete case, $I(X^n;Y^n) = H(Y^n) - H(Y^n|X^n)$ and the remainder of the proof follows as in the discrete case. Letting the quantizers become asymptotically accurate then completes the proof. □

Chapter 9
Distortion and Information

Abstract A pair random process $(X, Y) = \{X_n, Y_n\}$ can be generated by a source and a channel, a source and a code, a combination of a source, channel, encoder, and decoder, or two sources and a coupling. We have developed in some detail the properties of two quantities characterizing relations between the two components of a pair process: the average distortion between the components and their mutual information rate. In this chapter relations are developed between distortion and rate, where rate is measured by mutual information. The primary results concern the Shannon distortion-rate function and its dual, the Shannon rate-distortion function, which will be seen to provide bounds on the relationships between distortion and entropy and to characterize the optimal performance in source coding and rate-constrained simulation systems.

9.1 The Shannon Distortion-Rate Function

Given a source $[A, \mu]$ and a fidelity criterion ρ_n; $n = 1, 2, \ldots$ defined on $A \times \hat{A}$, where $\hat{A}$ is the *reproduction alphabet*, the Shannon distortion-rate function (DRF) is defined in terms of a nonnegative parameter called *rate* by

$$D(R, \mu) = \limsup_{N \to \infty} \frac{1}{N} D_N(R, \mu^N)$$

where

$$D_N(R, \mu^N) = \inf_{p^N \in \mathcal{R}_N(R, \mu^N)} E_{p^N} \rho_N(X^N, Y^N)$$

where $\mathcal{R}_N(R, \mu^N)$ is the collection of all distributions p^N for the coordinate random vectors X^N and Y^N on the space $(A^N \times \hat{A}^N, \mathcal{B}_A^N \times \mathcal{B}_{\hat{A}}^N)$ with the properties that

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(1) p^N induces the given marginal μ^N; that is, $p^N(F \times \hat{A}^N) = \mu^N(F)$ for all $F \in \mathcal{B}_A^N$, and
(2) the mutual information satisfies

$$\frac{1}{N} I_{p^N}(X^N; \hat{X}^N) \le R.$$

D_N is called the *Nth order distortion-rate function*. If $\mathcal{R}_N(R, \mu^N)$ is empty, then $D_N(R, \mu^N)$ is ∞.

Readers familiar with the rate-distortion theory literature may notice that the definition is not the one usually encountered, which is an infimum over N and not a limit supremum. This is simply because the infimum is most useful for stationary processes, while the limit supremum turns out to be the right form in the general AMS case eventually considered here. It will be seen shortly that the two definitions are equal if the source is stationary.

Various other notations are used for the distortion-rate function in the literature and here when convenience suggests it. The distribution μ may be dropped if it is fixed, writing $D(R)$ instead of $D(R, \mu)$. Sometimes the random variable is used instead of the distribution, e.g., writing $D_X(R)$ for $D(R, \mu)$.

An alternative approach to minimizing over joint distributions with a constrained input marginal is to minimize over *test channels* or regular conditional probabilities ν which induce a joint distribution by the hookup $\mu\nu$. This is equivalent since we are considering only standard alphabets and hence any joint distribution p with input marginal distribution μ will induce a regular conditional probability distribution ν for which $p = \mu\nu$.

One can also define the dual or inverse function to the distortion-rate function, Shannon's rate-distortion function, by

$$R(D, \mu) = \limsup_{N \to \infty} \frac{1}{N} R_N(D, \mu^N) \tag{9.1}$$

$$R_N(D, \mu^N) = \inf_{p^N \in \mathcal{D}_N(D, \mu^N)} I_{p^N}(X^N; \hat{X}^N), \tag{9.2}$$

where $\mathcal{D}_N(D, \mu^N)$ is the collection of all distributions p^N for the coordinate random vectors X^N and Y^N on the space $(A^N \times \hat{A}^N$ with the properties that

(1) p^N induces the given marginal μ^N; that is, $p^N(F \times \hat{A}^N) = \mu^N(F)$ for all $F \in \mathcal{B}_A^N$, and
(2) the average distortion satisfies

$$\frac{1}{N} E_{p^N} \rho_N(X^N, Y^N) \le D.$$

In his original development of source coding subject to a fidelity criterion or rate-distortion theory, Shannon considered the rate-distortion function rather than the distortion-rate function [162, 163]. We emphasize the distortion-rate function because it is a better match to the formulation of source coding considered here. In particular, in communications applications it is usually the rate that is constrained by a communications or storage medium such as a noisy channel or a limited memory. The theory and algorithms for the evaluation of rate-distortion tradeoffs are usually simpler if stated using the rate-distortion viewpoint rather than the distortion-rate viewpoint, and most of the evaluation literature uses the rate-distortion approach. Hence we shall focus on the distortion-rate function for most of the development, the corresponding properties for rate-distortion functions follow similarly. For the discussion of bounding and evaluating the functions, we shall use the rate-distortion viewpoint.

9.2 Basic Properties

Lemma 9.1. *$D_N(R,\mu)$ and $D(R,\mu)$ are nonnegative, nonincreasing, and convex $\bigcup$ functions of R and hence are continuous in R for $R > 0$.*

Proof: Nonnegativity is obvious from the nonnegativity of distortion. Nonincreasing follows since if $R_2 > R_1$, then $\mathcal{R}_N(R_1,\mu^N) \subset \mathcal{R}_N(R_2,\mu^N)$ and hence a minimization over the larger set can not yield a worse (larger) result. Suppose that $p_i \in \mathcal{R}_N(R_i,\mu^N)$; $i = 1,2$ yields

$$E_{p_i}\rho_N(X^N,Y^N) \le D_N(R_i,\mu) + \epsilon.$$

From Lemma 7.19 mutual information is a convex $\bigcup$ function of the conditional distribution and hence if $\overline{p} = \lambda p_1 + (1-\lambda)p_2$, then

$$I_{\overline{p}} \le \lambda I_{p_1} + (1-\lambda)I_{p_2} \le \lambda R_1 + (1-\lambda)R_2$$

and hence $\overline{p} \in \mathcal{R}_N(\lambda R_1 + (1-\lambda)R_2)$ and therefore

$$\begin{aligned} D_N(\lambda R_1 + (1-\lambda)R_2) &\le E_{\overline{p}}\rho_N(X^N,Y^N) \\ &= \lambda E_{p_1}\rho_N(X^N,Y^N) + (1-\lambda)E_{p_2}\rho_N(X^N,Y^N) \\ &\le \lambda D_N(R_1,\mu) + (1-\lambda)D_N(R_2,\mu). \end{aligned}$$

Since $D(R,\mu)$ is the limit of $D_N(R,\mu)$, it too is nonincreasing and convex. It is well known from real analysis that convex functions are continuous except possibly at their end points. □

The following lemma shows that when the underlying source is stationary and the fidelity criterion is subadditive (e.g., additive), then the limit defining $D(R,\mu)$ is an infimum.

Lemma 9.2. *If the source μ is stationary and the fidelity criterion is subadditive, then*

$$D(R,\mu) = \lim_{N\to\infty} D_N(R,\mu) = \inf_N \frac{1}{N} D_N(R,\mu).$$

Proof: Fix N and $n < N$ and let $p^n \in \mathcal{R}_n(R,\mu^n)$ yield

$$E_{p^n}\rho_n(X^n,Y^n) \le D_n(R,\mu^n) + \frac{\epsilon}{2}$$

and let $p^{N-n} \in \mathcal{R}_{N-n}(R,\mu^{N-n})$ yield

$$E_{p^{N-n}}\rho_{N-n}(X^{N-n},Y^{N-n}) \le D_{N-n}(R,\mu^{N-n}) + \frac{\epsilon}{2}.$$

p_n together with μ^n implies a regular conditional probability $q(F|x^n)$, $F \in \mathcal{B}_{\hat{A}}^n$. Similarly p_{N-n} and μ^{N-n} imply a regular conditional probability $r(G|x^{N-n})$. Define now a regular conditional probability $t(\cdot|x^N)$ by its values on rectangles as

$$t(F\times G|x^N) = q(F|x^n)r(G|x_n^{N-n});\ F \in \mathcal{B}_{\hat{A}}^n, G \in \mathcal{B}_{\hat{A}}^{N-n}.$$

Note that this is the finite dimensional analog of a block memoryless channel with two blocks. Let $p^N = \mu^N t$ be the distribution induced by μ and t. Then exactly as in Lemma 8.8 we have because of the conditional independence that

$$I_{p^N}(X^N;Y^N) \le I_{p^N}(X^n;Y^n) + I_{p^N}(X_n^{N-n};Y_n^{N-n})$$

and hence from stationarity

$$\begin{aligned} I_{p^N}(X^N;Y^N) &\le I_{p^n}(X^n;Y^n) + I_{p^{N-n}}(X^{N-n};Y^{N-n}) \\ &\le nR + (N-n)R = NR \end{aligned}$$

so that $p^N \in \mathcal{R}_N(R,\mu^N)$. Thus

$$\begin{aligned} D_N(R,\mu^N) &\le E_{p^N}\rho_N(X^N,Y^N) \\ &\le E_{p^N}\left(\rho_n(X^n,Y^n) + \rho_{N-n}(X_n^{N-n},Y_n^{N-n})\right) \\ &= E_{p^n}\rho_n(X^n,Y^n) + E_{p^{N-n}}\rho_{N-n}(X^{N-n},Y^{N-n}) \\ &\le D_n(R,\mu^n) + D_{N-n}(R,\mu^{N-n}) + \epsilon. \end{aligned}$$

Thus since ϵ is arbitrary we have shown that if $d_n = D_n(R,\mu^n)$, then

$$d_N \le d_n + d_{N-n};\ n \le N;$$

that is, the sequence d_n is subadditive. The lemma then follows immediately from the convergence of subadditive functions (e.g., Lemma 7.5.1 of [55] or Lemma 8.5.3 of [58]). □

IID Sources

If the source is IID, then the evaluation of the distortion-rate function becomes particularly simple.

Lemma 9.3. *If a source μ is IID, then $N^{-1}D_N(R,\mu^N) = D_1(R,\mu^1) = D(R,\mu)$ for all N.*

Proof. Suppose that the distribution p^N for (X^N, Y^N) approximately yields $D_N(R,\mu^N)$, that is, p^N has μ^N as input marginal, $I(p^N) \le NR$, and

$$E_{p^N}\left[d_N(X^N, Y^N)\right] \le D_N(R,\mu^N) + \epsilon$$

for small $\epsilon > 0$. Let p_i denote the induced distribution for (X_i, Y_i). Since X is IID, all the p_i have μ as input marginal. From Lemma 8.8, since μ is IID

$$NR \ge I(p^N) = I(X^N; Y^N)) \ge \sum_{i=0}^{N-1} I(X_i; Y_i) = \sum_{i=0}^{N-1} I(p_i)$$

which implies with the convexity of $D_1(R,\mu)$ in R that

$$\begin{aligned}\frac{1}{N}(D_N(R,\mu^N) + \epsilon) &\ge \frac{1}{N}E_{p^N}\left[d_N(X^N, Y^N)\right] \\ &= \frac{1}{N}\sum_{i=0}^{N-1} E_{p_i}\left[d(X_i, Y_i)\right] \ge \frac{1}{N}\sum_{i=0}^{N-1} D_1(I(p_i),\mu) \\ &\ge D_1(\frac{1}{N}\sum_{i=0}^{N-1} I(p_i),\mu) \ge D_1(R,\mu),\end{aligned}$$

where the final step used the fact that $D_1(R,\mu)$ is nonincreasing in its argument. This proves that $D_N(R,\mu) \ge ND_1(R,\mu)$. If p^1 approximately achieves $D_1(R,\mu)$ in the sense that $I(p^1) \le R$ and $E_{p^1}\left[d(X_0, Y_0)\right] \le D_1(R,\mu) + \epsilon$, then from Lemma 8.8$I(p^N) = NI(p_i) \le NR$ and hence

$$\begin{aligned}D_N(R,\mu) &\le E_{p^N}\left[\rho_N(X^N, Y^N)\right] \\ &= NE_{p^1}\left[\rho(X_0, Y_0)\right] \le N[D_1(R,\mu^1) + \epsilon]\end{aligned}$$

which implies that $D_N(R,\mu) \le ND_1(R,\mu^1)$ since ϵ is arbitrary. Thus $D_N(R,\mu) = ND_1(R,\mu^1)$ □

The corresponding result is true for the rate-distortion function.

9.3 Process Definitions of the Distortion-Rate Function

As with the $\overline{\rho}$ distance, there are alternative characterizations of the distortion-rate function when the process is stationary. The remainder of this section is devoted to developing these results. The idea of a stationarized block memoryless (SBM) channel will play an important role in relating nth order distortion-rate functions to the process definitions. We henceforth assume that the input source μ is stationary and we confine interest to additive fidelity criteria based on a per-letter distortion $\rho = \rho_1$.

The basic process DRF is defined by

$$\overline{D}_s(R,\mu) = \inf_{p \in \overline{\mathcal{R}}_s(R,\mu)} E_p \rho(X_0, Y_0),$$

where $\overline{\mathcal{R}}_s(R,\mu)$ is the collection of all stationary processes p having μ as an input distribution and having mutual information rate $\overline{I}_p = \overline{I}_p(X;Y) \le R$. The original idea of a process rate-distortion function was due to Kolmogorov and his colleagues [101] [49] (see also [23]). The idea was later elaborated by Marton [119], Gray, Neuhoff, and Omura [63], and Hashimoto [76].

Recalling that the L^1 ergodic theorem for information density holds when $\overline{I}_p = I_p^*$; that is, the two principal definitions of mutual information rate yield the same value, we also define the process DRF

$$D_s^*(R,\mu) = \inf_{p \in \mathcal{R}_s^*(R,\mu)} E_p \rho(X_0, Y_0),$$

where $\mathcal{R}_s^*(R,\mu)$ is the collection of all stationary processes p having μ as an input distribution, having mutual information rate $\overline{I}_p \le R$, and having $\overline{I}_p = I_p^*$. If μ is both stationary and ergodic, define the corresponding ergodic process DRF's by

$$\overline{D}_e(R,\mu) = \inf_{p \in \overline{\mathcal{R}}_e(R,\mu)} E_p \rho(X_0, Y_0),$$

$$D_e^*(R,\mu) = \inf_{p \in \mathcal{R}_e^*(R,\mu)} E_p \rho(X_0, Y_0),$$

where $\overline{\mathcal{R}}_e(R,\mu)$ is the subset of $\overline{\mathcal{R}}_s(R,\mu)$ containing only ergodic measures and $\mathcal{R}_e^*(R,\mu)$ is the subset of $\mathcal{R}_s^*(R,\mu)$ containing only ergodic measures.

Theorem 9.1. *Given a stationary source which possesses a reference letter in the sense that there exists a letter $a^* \in \hat{A}$ such that*

$$E_\mu \rho(X_0, a^*) \le \rho^* < \infty. \tag{9.3}$$

Fix $R > 0$. If $D(R,\mu) < \infty$, then

$$D(R,\mu) = \overline{D}_s(R,\mu) = D_s^*(R,\mu).$$

If in addition μ is ergodic, then also

$$D(R,\mu) = \overline{D}_e(R,\mu) = D_e^*(R,\mu).$$

The proof of the theorem depends strongly on the relations among distortion and mutual information for vectors and for SBM channels. These are stated and proved in the following lemma, the proof of which is straightforward but somewhat tedious. The theorem is proved after the lemma.

Lemma 9.4. *Let μ be the process distribution of a stationary source $\{X_n\}$. Let $\rho_n;\ n = 1, 2, \cdots$ be a subadditive (e.g., additive) fidelity criterion. Suppose that there is a reference letter $a^* \in \hat{A}$ for which (9.3) holds. Let p^N be a measure on $(A^N \times \hat{A}^N, \mathcal{B}_A^N \times \mathcal{B}_{\hat{A}}^N)$ having μ^N as input marginal; that is, $p^N(F \times \hat{A}^N) = \mu^N(F)$ for $F \in \mathcal{B}_A^N$. Let q denote the induced conditional probability measure; that is, $q_{x^N}(F)$, $x^N \in A^N$, $F \in \mathcal{B}_{\hat{A}}^N$, is a regular conditional probability measure. (This exists because the spaces are standard.) We abbreviate this relationship as $p^N = \mu^N q$. Let X^N, Y^N denote the coordinate functions on $A^N \times \hat{A}^N$ and suppose that*

$$E_{p^N} \frac{1}{N} \rho_N(X^N, Y^N) \le D \tag{9.4}$$

and

$$\frac{1}{N} I_{p^N}(X^N; Y^N) \le R. \tag{9.5}$$

If ν is an (N, δ) SBM channel induced by q as in Section 2.14 and if $p = \mu\nu$ is the resulting hookup and $\{X_n, Y_n\}$ the input/output pair process, then

$$\frac{1}{N} E_p \rho_N(X^N, Y^N) \le D + \rho^* \delta \tag{9.6}$$

and

$$\bar{I}_p(X;Y) = I_p^*(X;Y) \le R; \tag{9.7}$$

that is, the resulting mutual information rate of the induced stationary process satisfies the same inequality as the vector mutual information and the resulting distortion approximately satisfies the vector inequality provided δ is sufficiently small. Observe that if the fidelity criterion is additive, the (9.6) becomes

$$E_p \rho_1(X_0, Y_0) \le D + \rho^* \delta.$$

Proof: We first consider the distortion as it is easier to handle. Since the SBM channel is stationary and the source is stationary, the hookup p is stationary and

$$\frac{1}{n}E_p\rho_n(X^n,Y^n) = \frac{1}{n}\int dm_Z(z)E_{p_z}\rho_n(X^n,Y^n),$$

where p_z is the conditional distribution of $\{X_n,Y_n\}$ given $\{Z_n\}$, the punctuation process of Section 2.14. Note that the above formula reduces to $E_p\rho(X_0,Y_0)$ if the fidelity criterion is additive because of the stationarity. Given z, define $J_0^n(z)$ to be the collection of indices of z^n for which z_i is not in an N-cell. (See the discussion in Section 2.14.) Let $J_1^n(z)$ be the collection of indices for which z_i begins an N-cell. If we define the event $G = \{z : z_0 \text{ begins an } N-\text{cell}\}$, then $i \in J_1^n(z)$ if $T^i z \in G$. From Corollary 2.2 $m_Z(G) \le N^{-1}$. Since μ is stationary and $\{X_n\}$ and $\{Z_n\}$ are mutually independent,

$$\begin{aligned} nE_{p_z}\rho_n(X^n,Y^n) &\le \sum_{i\in J_0^n(z)} E_{p_z}\rho(X_i,a^*) + N\sum_{i\in J_1^n(z)} E_{p_z}\rho(X_i^N,Y_i^N) \\ &= \sum_{i=0}^{n-1} 1_{G^c}(T^i z)\rho^* + \sum_{i=0}^{n-1} E_{p^N}\rho_N 1_G(T^i z). \end{aligned}$$

Since m_Z is stationary, integrating the above we have that

$$E_p\rho_1(X_0,Y_0) = \rho^* m_Z(G^c) + Nm_Z(G)E_{p^N}\rho_N \le \rho^*\delta + E_{p^N}\rho_N,$$

proving (9.6).

Let r_m and t_m denote asymptotically accurate quantizers on A and $\hat{A}$; that is, as in Corollary 8.2 define

$$\hat{X}^n = r_m(X)^n = (r_m(X_0),\cdots,r_m(X_{n-1}))$$

and similarly define $\hat{Y}^n = t_m(Y)^n$. Then

$$I(r_m(X)^n;t_m(Y)^n) \underset{m\to\infty}{\to} I(X^n;Y^n)$$

and

$$\bar{I}(r_m(X);t_m(Y)) \underset{m\to\infty}{\to} I^*(X;Y).$$

We wish to prove that

$$\begin{aligned} \bar{I}(X;Y) &= \lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{n}I(r_m(X)^n;t_m(Y)^n) \\ &= \lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{n}I(r_m(X)^n;t_m(Y)^n) \\ &= I^*(X;Y) \end{aligned}$$

Since $\overline{I} \geq I^*$, we must show that

$$\lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{n}I(r_m(X)^n;t_m(Y)^n) \leq \lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{n}I(r_m(X)^n;t_m(Y)^n).$$

We have that

$$I(\hat{X}^n;\hat{Y}^n) = I((\hat{X}^n,Z^n);\hat{Y}^n) - I(Z^n,\hat{Y}^n|\hat{X}^n)$$

and

$$I((\hat{X}^n,Z^n);\hat{Y}^n) = I(\hat{X}^n;\hat{Y}^n|Z^n) + I(\hat{Y}^n;Z^n) = I(\hat{X}^n;\hat{Y}^n|Z^n)$$

since $\hat{X}^n$ and Z^n are independent. Similarly,

$$\begin{aligned} I(Z^n;\hat{Y}^n|\hat{X}^n) &= H(Z^n|\hat{X}^n) - H(Z^n|\hat{X}^n,\hat{Y}^n) \\ &= H(Z^n) - H(Z^n|\hat{X}^n,\hat{Y}^n) = I(Z^n;(\hat{X}^n,\hat{Y}^n)). \end{aligned}$$

Thus we need to show that

$$\begin{aligned} &\lim_{n\to\infty}\lim_{m\to\infty}\left(\frac{1}{n}I(r_m(X)^n;t_m(Y)^n|Z^n) - \frac{1}{n}I(Z^n,(r_m(X)^n,t_m(Y)^n))\right) \leq \\ &\lim_{m\to\infty}\lim_{n\to\infty}\left(\frac{1}{n}I(r_m(X)^n;t_m(Y)^n|Z^n) - \frac{1}{n}I(Z^n,(r_m(X)^n,t_m(Y)^n))\right). \end{aligned}$$

Since Z_n has a finite alphabet, the limits of $n^{-1}I(Z^n,(r_m(X)^n,t_m(Y)^n))$ are the same regardless of the order from Theorem 8.2. Thus $\overline{I}$ will equal I^* if we can show that

$$\begin{aligned} \overline{I}(X;Y|Z) &= \lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{n}I(r_m(X)^n;t_m(Y)^n|Z^n) \\ &\leq \lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{n}I(r_m(X)^n;t_m(Y)^n|Z^n) = I^*(X;Y|Z).. \quad (9.8) \end{aligned}$$

This we now proceed to do. From Lemma 7.21 we can write

$$I(r_m(X)^n;t_m(Y)^n|Z^n) = \int I(r_m(X)^n;t_m(Y)^n|Z^n = z^n)\,dP_{Z^n}(z^n).$$

Abbreviate $I(r_m(X)^n;t_m(Y)^n|Z^n = z^n)$ to $I_z(\hat{X}^n;\hat{Y}^n)$. This is simply the mutual information between $\hat{X}^n$ and $\hat{Y}^n$ under the distribution for $(\hat{X}^n,\hat{Y}^n)$ given a particular random blocking sequence z. We have that

$$I_z(\hat{X}^n;\hat{Y}^n) = H_z(\hat{Y}^n) - H_z(\hat{Y}^n|\hat{X}^n).$$

Given z, let $J_0^n(z)$ be as before. Let $J_2^n(z)$ denote the collection of all indices i of z_i for which z_i begins an N cell *except* for the final such index (which may begin an N-cell not completed within z^n). Thus $J_2^n(z)$

is the same as $J_1^n(z)$ except that the largest index in the latter collection may have been removed if the resulting N-cell was not completed within the n-tuple. We have using standard entropy relations that

$$I_z(\hat{X}^n;\hat{Y}^n) \ge \sum_{i\in J_0^n(z)} \left(H_z(\hat{Y}_i|\hat{Y}^i) - H_z(\hat{Y}_i|\hat{Y}^i,\hat{X}^{i+1})\right) + \sum_{i\in J_2^n(z)} \left(H_z(\hat{Y}_i^N|\hat{Y}^i) - H_z(\hat{Y}_i^N|\hat{Y}^i,\hat{X}^{i+N})\right). \quad (9.9)$$

For $i \in J_0^n(z)$, however, Y_i is a^* with probability one and hence

$$H_z(\hat{Y}_i|\hat{Y}^i) \le H_z(\hat{Y}_i) \le H_z(Y_i) = 0$$

and

$$H_z(\hat{Y}_i|\hat{Y}^i,\hat{X}^{i+1}) \le H_z(\hat{Y}_i) \le H_z(Y_i) = 0.$$

Thus we have the bound

$$\begin{aligned} I_z(\hat{X}^n;\hat{Y}^n) &\ge \sum_{i\in J_2^n(z)} \left(H_z(\hat{Y}_i^N|\hat{Y}^i) - H_z(\hat{Y}_i^N|\hat{Y}^i,\hat{X}^{i+N})\right). \\ &= \sum_{i\in J_2^n(z)} \left(I_z(\hat{Y}_i^N;(\hat{Y}^i,\hat{X}^i+N)) - I_z(\hat{Y}_i^N;\hat{Y}^i)\right) \\ &\ge \sum_{i\in J_2^n(z)} \left(I_z(\hat{Y}_i^N;\hat{X}_i^N) - I_z(\hat{Y}_i^N;\hat{Y}^i)\right), \end{aligned} \quad (9.10)$$

where the last inequality follows from the fact that $I(U;(V,W)) \ge I(U;V)$.

For $i \in J_2^n(z)$ we have by construction and the stationarity of μ that

$$I_z(\hat{X}_i^N;\hat{Y}_i^N) = I_{p^N}(\hat{X}^N;\hat{Y}^N). \quad (9.11)$$

As before let $G = \{z : z_0 \text{ begins an } N-\text{cell}\}$. Then $i \in J_2^n(z)$ if $T^i z \in G$ and $i < n - N$ and we can write

$$\frac{1}{n} I_z(\hat{X}^n;\hat{Y}^n) \ge \frac{1}{n} I_{p^N}(\hat{X}^N;\hat{Y}^N) \sum_{i=0}^{n-N-1} 1_G(T^i z) - \frac{1}{n} \sum_{i=0}^{n-N-1} I_z(\hat{Y}_i^N;\hat{Y}^i) 1_G(T^i z).$$

All of the above terms are measurable functions of z and are nonnegative. Hence they are integrable (although we do not yet know if the integral is finite) and we have that

$$\frac{1}{n}I(\hat{X}^n;\hat{Y}^n) \ge$$

$$I_{p^n}(\hat{X}^N;\hat{Y}^N)m_Z(G)\frac{n-N}{n} - \frac{1}{n}\sum_{i=0}^{n-N-1}\int dm_Z(z)I_z(\hat{Y}_i^N;\hat{Y}^i)1_G(T^i z).$$

To continue we use the fact that since the processes are stationary, we can consider it to be a two-sided process (if it is one-sided, we can imbed it in a two-sided process with the same probabilities on rectangles). By construction

$$I_z(\hat{Y}_i^N;\hat{Y}^i) = I_{T^i z}(\hat{Y}_0^N;(Y_{-i},\cdots,Y_{-1}))$$

and hence since m_Z is stationary we can change variables to obtain

$$\frac{1}{n}I(\hat{X}^n;\hat{Y}^n) \ge I_{p^n}(\hat{X}^N;\hat{Y}^N)m_Z(G)\frac{n-N}{n}$$
$$- \frac{1}{n}\sum_{i=0}^{n-N-1}\int dm_Z(z)I_z(\hat{Y}_0^N;(\hat{Y}_{-i},\cdots,\hat{Y}_{-1}))1_G(z).$$

We obtain a further bound from the inequalities

$$I_z(\hat{Y}_0^N;(\hat{Y}_{-i},\cdots,\hat{Y}_{-1})) \le I_z(Y_0^N;(Y_{-i},\cdots,Y_{-1})) \le I_z(Y_0^N;Y^-)$$

where $Y^- = (\cdots,Y_{-2},Y_{-1})$. Since $I_z(Y_0^N;Y^-)$ is measurable and nonnegative, its integral is defined and hence

$$\lim_{n\to\infty}\frac{1}{n}I(\hat{X}^n;\hat{Y}^n|Z^n) \ge I_{p^n}(\hat{X}^N;\hat{Y}^N)m_Z(G) - \int_G dm_Z(z)I_z(Y_0^N;Y^-).$$

We can now take the limit as $m \to \infty$ to obtain

$$I^*(X;Y|Z) \ge I_{p^n}(X^N;Y^N)m_Z(G) - \int_G dm_Z(z)I_z(Y_0^N;Y^-). \tag{9.12}$$

This provides half of what we need.

Analogous to (9.9) we have the upper bound

$$I_z(\hat{X}^n;\hat{Y}^n) \le \sum_{i\in J_1^n(z)}\left(I_z(\hat{Y}_i^N;(\hat{Y}^i,\hat{X}^{i+N})) - I_z(\hat{Y}_i^N;\hat{Y}^i)\right). \tag{9.13}$$

We note in passing that the use of J_1 here assumes that we are dealing with a one-sided channel and hence there is no contribution to the information from any initial symbols not contained in the first N-cell. In the two-sided case time 0 could occur in the middle of an N-cell and one could fix the upper bound by adding the first index less than 0 for which z_i begins an N-cell to the above sum. This term has no affect on the limits. Taking the limits as $m \to \infty$ using Lemma 7.14 we have that

$$I_z(X^n;Y^n) \le \sum_{i\in J_1^n(z)} \left(I_z(Y_i^N;(Y^i,X^{i+N})) - I_z(Y_i^N;Y^i)\right).$$

Given $Z^n = z^n$ and $i \in J_1^n(z)$, $(X^i,Y^i) \to X_i^N \to Y_i^N$ forms a Markov chain because of the conditional independence and hence from Lemma 7.15 and Corollary 7.14

$$I_z(Y_i^N,(Y^i,X^{i+N})) = I_z(X_i^N;Y_i^N) = I_{p^N}(X^N;Y^N).$$

Thus we have the upper bound

$$\frac{1}{n}I_z(X^n;Y^n) \le \frac{1}{n}I_{p^N}(X^N;Y^N)\sum_{i=0}^{n-1} 1_G(T^iz) - \frac{1}{n}\sum_{i=0}^{n-1} I_z(Y_i^N;Y^i)1_G(T^iz).$$

Taking expectations and using stationarity as before we find that

$$\frac{1}{n}I(X^n;Y^n|Z^n) \le$$
$$I_{p^N}(X^N;Y^N)m_Z(G) - \frac{1}{n}\sum_{i=0}^{n-1}\int_G dm_Z(z)I_z(Y_0^N;(Y_{-i},\cdots,Y_{-1})).$$

Taking the limit as $n \to \infty$ using Lemma 7.22 yields

$$\overline{I}(X;Y|Z) \le I_{p^N}(X^N;Y^N)m_Z(G) - \int_G dm_Z(z)I_z(Y_0^N;Y^-). \tag{9.14}$$

Combining this with (9.12) proves that $\overline{I}(X;Y|Z) \le I^*(X;Y|Z)$ and hence that $\overline{I}(X;Y) = I^*(X;Y)$. It also proves that

$$\begin{aligned}\overline{I}(X;Y) &= \overline{I}(X;Y|Z) - \overline{I}(Z;(X,Y)) \le \overline{I}(X;Y|Z)\\ &\le I_{p^N}(X^N;Y^N)m_Z(G) \le \frac{1}{N}I_{p^N}(X^N;Y^N)\end{aligned}$$

using Corollary 2.2 to bound $m_X(G)$. This proves (9.7). □

Proof of the theorem: We have immediately that

$$\mathcal{R}_e^*(R,\mu) \subset \mathcal{R}_s^*(R,\mu) \subset \overline{\mathcal{R}}_s(R,\mu)$$

and

$$\mathcal{R}_e^*(R,\mu) \subset \overline{\mathcal{R}}_e(R,\mu) \subset \overline{\mathcal{R}}_s(R,\mu),$$

and hence we have for stationary sources that

$$\overline{D}_s(R,\mu) \le D_s^*(R,\mu) \tag{9.15}$$

and for ergodic sources that

$$\overline{D}_s(R,\mu) \le D_s^*(R,\mu) \le D_e^*(R,\mu) \tag{9.16}$$

and

$$\overline{D}_s(R,\mu) \le \overline{D}_e(R,\mu) \le D_e^*(R,\mu). \tag{9.17}$$

We next prove that

$$\overline{D}_s(R,\mu) \ge D(R,\mu). \tag{9.18}$$

If $\overline{D}_s(R,\mu)$ is infinite, the inequality is obvious. Otherwise fix $\epsilon > 0$ and choose a $p \in \overline{\mathcal{R}}_s(R,\mu)$ for which $E_p\rho_1(X_0,Y_0) \le \overline{D}_s(R,\mu) + \epsilon$ and fix $\delta > 0$ and choose m so large that for $n \ge m$ we have that

$$n^{-1}I_p(X^n;Y^n) \le \overline{I}_p(X;Y) + \delta \le R + \delta.$$

For $n \ge m$ we therefore have that $p^n \in \mathcal{R}_n(R+\delta,\mu^n)$ and hence

$$\overline{D}_s(R,\mu) + \epsilon = E_{p^n}\rho_n \ge D_n(R+\delta,\mu) \ge D(R+\delta,\mu).$$

From Lemma 9.1 $D(R,\mu)$ is continuous in R and hence (9.18) is proved.

Lastly, fix $\epsilon > 0$ and choose N so large and $p^N \in \mathcal{R}_N(R,\mu^N)$ so that

$$E_{p^N}\rho_N \le D_N(R,\mu^N) + \frac{\epsilon}{3} \le D(R,\mu) + \frac{2\epsilon}{3}.$$

Construct the corresponding (N,δ)-SBM channel as in Section 2.14 with δ small enough to ensure that $\delta\rho^* \le \epsilon/3$. Then from Lemma 9.2 we have that the resulting hookup p is stationary and that $\overline{I}_p = I_p^* \le R$ and hence $p \in \mathcal{R}_s^*(R,\mu) \subset \overline{\mathcal{R}}_s(R,\mu)$. Furthermore, if μ is ergodic then so is p and hence $p \in \mathcal{R}_e^*(R,\mu) \subset \overline{\mathcal{R}}_e(R,\mu)$. From Lemma 9.2 the resulting distortion is

$$E_p\rho_1(X_0,Y_0) \le E_{p^N}\rho_N + \rho^*\delta \le D(R,\mu) + \epsilon.$$

Since $\epsilon > 0$ this implies the exisitence of a $p \in \mathcal{R}_s^*(R,\mu)$ ($p \in \mathcal{R}_e^*(R,\mu)$ if μ is ergodic) yielding $E_p\rho_1(X_0,Y_0)$ arbitrarily close to $D(R,\mu$. Thus for any stationary source $D_s^*(R,\mu) \le D(R,\mu)$ and for any ergodic source $D_e^*(R,\mu) \le D(R,\mu)$.

With (9.15)–(9.18) this completes the proof. □

The previous lemma is technical but important in proving source coding theorems. It permits the construction of a stationary and ergodic pair process having rate and distortion near that of that for a finite dimensional vector described by the original source and a finite-dimensional conditional probability.

9.4 The Distortion-Rate Function as a Lower Bound

The Shannon distortion-rate function provides a simple lower bound to the performance in source coding systems and in constrained rate simulation systems. Both results will follow from from a simple inequality which we now develop.

Suppose that X is a stationary source with process distribution μ. Suppose also that X is encoded and then decoded in order to obtain a reproduction process $\hat{X}$, but we do not know the details of the code structures except that the resulting pair process $(X, \hat{X})$ with distribution p is AMS and the $\hat{X}$ process has a finite entropy rate $\overline{H}(\hat{X}) \le R$. Let η denote the distribution of $\hat{X}$. Let $\overline{p}$ denote the stationary mean with marginals $\overline{\mu} = \mu$, since μ is stationary, and $\overline{\eta}$. From Theorem 4.1,

$$\overline{H}(\hat{X}) = \overline{H}(\eta) = \overline{H}(\overline{\eta}). \tag{9.19}$$

For example, in a source coding system the codes might be stationary codes or block stationary codes, or block codes, or possibly even variable length codes, but the cascade of the operations must be AMS and yield a finite entropy rate reproduction. If, for example, there is a common finite alphabet for the output of the encoder and input to the decoder with 2^R letters, then since the decoder can not increase entropy rate, it must have entropy rate no greater than R. In the constrained simulation problem, the goal is to produce a process $\hat{X}$ as a coding of an IID process Z such that $\overline{H}(\hat{X}) \le R$ and the process $\hat{X}$ is as close as possible to X with respect to the $\overline{\rho}$ distance. The simulation problem was earlier formulated for stationary coding, but for the moment we allow other coding structures provided they yield an AMS process $\hat{X}$. We assume an additive fidelity criterion for which the single letter distortion is integrable with respect to the stationary mean (so that the fidelity criterion is convergent in the sense of Chapter 5). In this case the limiting distortion is

$$\rho_\infty = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \rho_1(X_i, \hat{X}_i) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \rho_1(X_i, \hat{X}_i)$$

where the limit exists $\overline{p}$-a.e. and p-a.e. and

$$\begin{aligned} \Delta(p) &= E_p \rho_\infty = E_{\overline{p}} \rho_1(X_0, \hat{X}_0) = \Delta(\overline{p}) \\ I^*(p) &= I^*(\overline{p}). \end{aligned}$$

From Lemma 8.4,

$$I^*(p) \le \overline{H}(\eta).$$

Putting all of this together, we have that

$$\Delta(p) = \Delta(\overline{p}) = E_{\overline{p}}\rho_1(X_0, \hat{X}_0)$$
$$I^*(p) = I^*(\overline{p}) \le R$$

and hence that

$$\Delta(p) \ge D_s(\mu, R). \tag{9.20}$$

The equation boils down to simply this: given any AMS pair process with a given performance with respect to a fidelity criterion and an entropy rate constraint on one component, then the average distortion can be no smaller that the stationary process distortion-rate function for the given constraint because the entropy rate of one component process overbounds the mutual information rate between the two components.

Corollary 9.1. *As in Section 5.4, consider a stationary source μ, a channel ν, and code classes $\mathcal{E}$, $\mathcal{D}$ for which if $f \in \mathcal{E}$, $g \in \mathcal{D}$, then the pair process $p_{X,\hat{X}}$ consisting of the input and output of the cascade $\mu f \nu g$ is AMS. Then the operational DRF of (5.14) is bound below by the stationary process Shannon DRF:*

$$\Delta(\mu, \nu, \mathcal{E}, \mathcal{D}) = \inf_{f\in\mathcal{E}, g\in\mathcal{D}} \Delta(\mu, f, \nu, g) \ge \overline{D}_s(\mu, \overline{H}(p_{\hat{X}})).$$

If there exists a reference letter and $\overline{H}(p_{\hat{X}}) < \infty$, then also $\Delta(\mu, \nu, \mathcal{E}, \mathcal{D}) \ge D(\mu, R)$.

For example, if the channel ν is noiseless with input alphabet A equal to the output alphabet, then $\overline{H}(p_{\hat{X}}) \le R = \log \|A\|$ and the bound becomes

$$\delta(R, \mu) = \Delta(\mu, \nu, \mathcal{E}, \mathcal{D}) \ge \overline{D}_s(R, \mu), \tag{9.21}$$

which is one form of the classic converse source coding theorem since from Theorem 9.1 the stationary process definition and the Shannon definitions are equal under the assumed conditions, that is, $\overline{D}_s(R, \mu) = D(\mu, R)$.

Consider next the constrained rate simulation problem of Section 6.7 of the best coding of an IID process Z with entropy rate $\overline{H}(Z)$. Suppose that $\hat{X}$ is a process with distribution $\mu_{\hat{X}}$ for which

$$\overline{\rho}(\mu_X, \mu_{\hat{X}}) \le \Delta_{X|Z} + \epsilon$$

for a small $\epsilon > 0$, which implies that there is a coupling π with marginals μ_X and $\mu_{\hat{X}}$ with $\overline{H}(\mu_{\hat{X}}) \le \overline{H}(Z)$ and distortion

$$\Delta(\pi) \le \Delta_{X|Z} + \epsilon$$

which from the lemma implies that

$$\Delta(\pi) \le D_s(\overline{H}(Z), \mu).$$

Since ϵ was arbitrary, we have the following corollary.

Corollary 9.2. *Given an IID process Z with entropy rate $\overline{H}(Z)$ and an additive fidelity criterion ρ_n and a stationary process X, then*

$$\Delta_{X|Z} \equiv \inf_f \rho_1(\mu_X, \mu_{f(Z)}) \geq D_s(\overline{H}(Z), \mu).$$

Thus the constrained rate problem also has the Shannon distortion-rate function as an unbeatable lower bound.

9.5 Evaluating the Rate-Distortion Function

The goal of this section is to consider an alternative characterization of the optimization that defines the RDF given by $R(D) = R_X(D) = R_N(D, \mu_X)$. The Shannon DRF and RDF are defined by information-theoretic optimization problems. Since average distortion is linear in terms of the joint distribution describing the random vectors and mutual information is a convex $\cup$ function of the conditional probability distribution of output given input — called the *test channel*, the techniques of convex optimization provide an approach to evaluating the DRF or RDF for specific sources and distortion measures of interest. Shannon [163] provided the first examples of evaluation of the RDF for memoryless sources with Hamming and squared-error distortion. Kolmogorov [101] considered the case of Gaussian vectors and processes with respect to a squared error distortion. Gallager [47] provided general Kuhn-Tucker conditions providing a variational approach to finding the RDF (see also Berger [11]). A key aspect of the variational approach is that the optimization over the test channel or pair distribution with constrained input marginal leads to an optimization of a reproduction distribution, the distribution of the output marginal of the pair distribution or source/test-channel hookup. Blahut [18] found an alternative formulation of the optimization in terms of relative entropy or divergence and an iterative algorithm for numerical solution, and Csiszár [25] extended these results and provided an elegant and rigorous development for general alphabets.

For the rest of this section, we consider the finite-order Shannon rate-distortion function (RDF) for vectors $X = X^N$ with distribution μ_X. We drop the superscripts for the dimension as it is assumed fixed. We will often drop the random variable subscript on a distribution if it is clear from context, so that $\mu = \mu_X$ throughout. We pause to summarize the notational shortcuts for this section.

We consider joint distributions $\pi_{X,Y}$ and $p_{X,Y}$ for a pair of random vectors (X, Y) with alphabet $(A_X \times A_Y, \mathcal{B}_{A_X} \times \mathcal{B}_{A_Y})$. Given a joint distribution π_{XY}, denote the induced marginal distributions by π_X and π_Y,

that is,

$$\pi_X(G) = \pi_{XY}(G \times A_Y); G \in \mathcal{B}_{A_X}$$
$$\pi_Y(G) = \pi_{XY}(A_X \times G); G \in \mathcal{B}_{A_Y}.$$

The marginals p_X and p_Y are similarly defined. If we are focusing only on joint distributions, the subscripts indicating the random variables will be dropped, that is, $\pi = \pi_{X,Y}$ and $p = p_{X,Y}$. Throughout this section $\pi_X = \mu_X = \mu$ and $p_X = \mu_X = \mu$, either by assumption or by demonstration. This constraint on the input distribution will be denoted by

$$\pi \in \mathcal{P}(\mu_X).$$

We will use η to denote a distribution on the reproduction Y, but it is not fixed. It is used simply as shorthand for an output marginal, which might be induced or optimized over. In particular, given a reproduction distribution η, we will construct a special joint distribution p which will be denoted p_η. This admittedly takes liberties with notation, but p_η will mean a joint distribution constructed using a given reproduction distribution η. The construction will be such that p_η need not have η as its marginal output distribution — η is simply used in the construction and the form of p_η depends on η.

We can express the RDF as

$$R(D) = \inf_{\pi_{XY}:\pi_{XY}\in\mathcal{P}(\mu_X),\rho(\pi_{XY})\le D} I(\pi_{XY}) \tag{9.22}$$

where

$$\rho(\pi_{XY}) = E_{\pi_{XY}}(\rho(X,Y)) = \int d\pi_{XY}(x,y)\rho(x,y) \tag{9.23}$$

$$I(\pi_{XY}) = I(X;Y) = \int d\pi_{XY}(x,y) \log \frac{\pi_{XY}(x,y)}{d(\pi_X \times \pi_Y)(x,y)} \tag{9.24}$$

For simplicity it is assumed that $R(D) < \infty$ for $D > 0$ and that $R(D) \to 0$ as $D \to \infty$. These assumptions reflect typical behavior and the details required for removing these simplifying assumptions may be found in Csiszár [25].

The most basic properties of the RDF parallel those for the DRF in Lemma 9.1 as summarized in the following lemma. The proof is omitted since it is a minor variation of the DRF case.

Lemma 9.5. *The Shannon RDF $R(D)$ is a nonnegative convex nonincreasing function of D.*

Note that convexity implies that $R(D)$ can not be a constant other than 0 over an interval of D given our assumption that $R(D)$ must go to zero as D grows.

Lemma 9.6. *An equivalent definition for the RDF is*

$$R(D) = \inf_{\pi_{XY}:\pi_{XY}\in\mathcal{P}(\mu_X),\rho(\pi_{XY})=D} I(\pi_{XY}); \tag{9.25}$$

that is, the inequality constraint can be replaced by an equality constraint.

Proof. Suppose the contrary and hence there is a $D_0 < D$ such that we can do better than D, that is for any $\epsilon > 0$ we can find a π with $D(\pi) < D_0$ and $I(\pi) \leq R(D) + \epsilon$. But then $I(\pi) \geq R(D_0)$ and hence, since ϵ is arbitrary, $R(D_0) \leq R(D)$. But $R(D)$ is nonincreasing in D and hence $R(D_0) = R(D)$, which violates the convexity of $R(D)$. □

The constrained optimization over all distributions is traditionally handled as an unconstrained minimization over distributions by focusing on the function

$$F(s) = \inf_{\pi_{XY}\in\mathcal{P}(\mu_X)} (I(\pi_{XY}) + s\rho(\pi_{XY}))\,; s \geq 0.$$

The function can be thought of as a variational or Lagrange multiplier formulation to remove the distortion constraint and incorporate it into the functional being minimized, but we will not use calculus to accomplish the minimization as was done in the original derivations. Instead we follow Csiszár's [26] approach and use the divergence inequality repeatedly to find conditions for global optimality.

We have easily that

$$\begin{aligned}
R(D) &= \inf_{\pi_{XY}:\pi_{XY}\in\mathcal{P}(\mu_X),\rho(\pi_{XY})\leq D} I(\pi_{XY}) \\
&= \inf_{\pi_{XY}:\pi_{XY}\in\mathcal{P}(\mu_X),\rho(\pi_{XY})\leq D} \left(I(\pi_{XY}) + s\rho(\pi_{XY}) \underbrace{-s\rho(\pi_{XY})}_{\geq -sD} \right) \\
&\geq \inf_{\pi_{XY}:\pi_{XY}\in\mathcal{P}(\mu_X),\rho(\pi_{XY})\leq D} (I(\pi_{XY}) + s\rho(\pi_{XY})) - sD \\
&\geq \inf_{\pi_{XY}:\pi_{XY}\in\mathcal{P}(\mu_X)} (I(\pi_{XY}) + s\rho(\pi_{XY})) - sD = F(s) - sD.
\end{aligned}$$

Thus for any fixed D,

$$R(D) + sD \geq F(s) \text{ for all } s \geq 0. \tag{9.26}$$

Consider a plot of $R(d)$ with rate on the vertical axis (the y axis) and distortion on the horizontal axis (the x axis). If s is fixed and d allowed to vary over nonnegative numbers, $F(s) - sd$ traces out a straight line $y = a - sx$ of slope $-s$ in the plot with vertical axis intercept $a = F(s)$. If we fix a value of $d = D$, then for *any* value of s we have seen that it must be true that $R(D) \geq F(s) - sD$. But $R(D)$ is a convex function, and hence at any point (D, R)=$(D, R(D))$ lying on the $R(D)$ curve there must exist a straight line passing through the point with no points above the $R(D)$

curve. Hence given a fixed $D > 0$, there must exist a slope $-s_D$ (perhaps more than one) and a straight line $y(x) = a - s_D x$ with slope $-s_D$ which passes through the point (D, R) such that $y(D) = a - s_D D = R(D)$ so that the y axis intercept of the straight line is at $a = R(D) + s_D D$. We say that such an s is *associated* with D. Since this is a tangent line and no points on the $R(D)$ curve can lie below it, for any D' we must have that

$$R(D') \geq y(D') = a - s_D D' = R(D) + s_D D - s_D D'$$

or, for all D'

$$R(D') + s_D D' \geq R(D) + s_D D. \tag{9.27}$$

Suppose that $\pi \in \mathcal{P}(\mu_X)$ approximately yields $F(s_D)$ so that for small $\epsilon > 0$ $I(\pi) + s_D \rho(\pi) \leq F(s) + \epsilon$. Then from (9.27) and (9.26)

$$F(s_D) + \epsilon \geq R(\rho(\pi)) + s_D \rho(\pi) \geq R(D) + s_D D \geq F(s_D), \tag{9.28}$$

which since ϵ can be made arbitrarily small implies

$$F(s_D) = R(D) + s_D D, \tag{9.29}$$

showing that the lower bound of (9.26) is achieved if s_D is associated with D.

Conversely, suppose that if instead of starting with D, we fix s, and π^* achieves a minimum in $F(s)$, that is,

$$F(s) = I(\pi^*) + s\rho(\pi^*) = \inf_{\pi \in \mathcal{P}(\mu_X)} (I(\pi) + s\rho(\pi)).$$

Define $D_s = \rho(\pi^*)$ and $R_s = I(\pi^*)$, then

$$F(s) = R_s + sD_s \geq \inf_{\pi \in \mathcal{P}(\mu_X); \rho(\pi) \leq D_s} I(\pi) + sD_s = R(D_s) + sD_s.$$

It is also true that

$$\begin{aligned} F(s) &= \inf_{\pi \in \mathcal{P}(\mu_X); \rho(\pi) \leq D_s} (I(\pi) + s\rho(\pi)) \\ &\leq \inf_{\pi \in \mathcal{P}(\mu_X); \rho(\pi) \leq D_s} (I(\pi) + sD_s) \\ &= \inf_{\pi \in \mathcal{P}(\mu_X); \rho(\pi) \leq D_s} I(\pi) + sD_s \\ &= R(D_s) + sD_s \end{aligned}$$

and hence $F(s) = R(D_s) + sD_s$, which means that s is associated with D_s and $R(D_s) = F(s) = sD_s$.

Summarizing the preceding development yields the following result.

Lemma 9.7.

$$R(D) = \max_{s \geq 0} (F(s) - sD).$$

The maximum is attained iff s is associated with D. If π achieves a minimum in $F(s)$, then s is associated with $D = \rho(\pi)$ and $R(D) = I(\pi)$.

The implication of the lemma is that the $R(D)$ curve can be thought of as being parametrized by s, the slope of the tangent to points on the curve. For each value of s we try to minimize $F(s)$ over all joint distributions π with marginal fixed by the source. If the minimizing distribution is found, its mutual information and average distortion yield a point on the rate-distortion curve. Thus the problem is to minimize $F(s)$. This topic is tackled next.

Recall from (7.28) the average mutual information can also be written as a divergence as

$$I(\pi_{XY}) = D(\pi_{XY} \| \pi_X \times \pi_Y),$$

so the problem is to find

$$F(s) = \inf_{\pi_{XY} \in \mathcal{P}(\mu_X)} \left(D(\pi_{XY} \| \pi_X \times \pi_Y) + s\rho(\pi_{XY}) \right).$$

A useful approach both for the mathematics of the solution and for suggesting an algorithm for computing the solution was introduced by Blahut [18]. The math and the algorithm have interesting parallels with the optimality properties of actual codes to be considered later and they provide an early example of an alternating optimization (AO) algorithm, an optimization that alternates between two steps, each of which optimizes one component of the function being optimized for the other [14]. To set up the method we make a change in the target function by introducing another distribution. This apparent complication will lead to several useful results. Define the functional

$$J(\pi_{XY}, \eta, s) = D(\pi_{XY} \| \pi_X \times \eta) + s\rho(\pi_{XY}) \tag{9.30}$$

and note that it differs from the function being optimized in $F(s)$ only by the replacement of the actual marginal π_Y in the divergence by a separate distribution η. Corollary 7.12 implies an immediate relation between the two:

$$D(\pi_{XY} \| \pi_X \times \pi_Y) = \inf_{\eta} D(\pi_{XY} \| \pi_X \times \eta). \tag{9.31}$$

If $D(\pi_{XY} \| \pi_X \times \eta)$ is finite, then π_{XY} will be absolutely continuous with respect to the product measure $\pi_X \times \eta$.

Given a distribution η, we construct a new joint distribution p_{XY} as follows. If $\rho(\pi_{XY})$ is finite, then $\{x, y : \rho(x, y) < \infty\}$ has positive probability and hence

$$\gamma_{\eta,s}(x) \equiv \frac{1}{\int e^{-\rho(x,y)} d\eta(y)} < \infty, \mu_X - \text{a.e.}.$$

Given a reproduction distribution η, define a joint distribution p_η by its density or Radon-Nikodym derivative with respect to the product measure $\mu_X \times \eta$ by

$$\frac{dp_\eta(x,y)}{d(\mu_X \times \eta)(x,y)} = \gamma_{\eta,s}(x)e^{-s\rho(x,y)}.$$

In other words, the distribution is specified by its values on rectangles as

$$p_\eta(F \times G) = \int_{F\times G} \gamma_{\eta,s}(x)e^{-s\rho(x,y)}d(\mu_X \times \eta)(x,y).$$

Intuitively, the density is constructed so that if the logarithm is taken, the result is the negative average distortion multiplied by s plus a normalization term. This will be shortly seen to be useful in expressing $J(\pi_{XY}, \eta, s)$ as a combination of simple terms. The X marginal of $p = p_\eta$ is easily found using Fubini's theorem to be

$$\begin{aligned}
p_X(F) &= p_\eta(F \times A_Y) \\
&= \int_{F\times A_Y} \frac{dp_\eta(x,y)}{d(\mu_X \times \eta)(x,y)} d(\mu_X \times \eta)(x,y) \\
&= \int_{F\times A_Y} \gamma_{\eta,s}(x)e^{-s\rho(x,y)}d(\mu_X \times \eta)(x,y) \\
&= \int_F d\mu_X(x) \left(\gamma_{\eta,s}(x) \int e^{-s\rho(x,y)}d\eta(y)\right) \\
&= \int_F d\mu_X(x) = \mu_X(F),
\end{aligned}$$

the source distribution. The output distribution p_Y, however, is not easily found in general and, perhaps surprisingly, need not equal η.

The introduction of the additional distribution η and the construction of the implied joint distribution p_η allows the following representations of the functional $J(\pi_{XY}, \eta, s)$.

Lemma 9.8. *The functional $J(\pi_{XY}, \eta, s) = D(\pi_{XY}\|\pi_X \times \eta) + s\rho(\pi_{XY})$ can be expressed as*

$$\begin{aligned}
J(\pi_{XY}, \eta, s) &= J(\pi_{XY}, \pi_Y, s) + D(\pi_Y\|\eta) && (9.32)\\
&= \int d\mu_X(x) \log \gamma_{\eta,s}(x) + D(\pi_{XY}\|p_\eta) && (9.33)\\
&= J(p_\eta, \eta, s) + D(\pi_{XY}\|p_\eta). && (9.34)
\end{aligned}$$

Proof. Csiszár [26] observes in his Lemma 1.3 that the equalities follow from the chain rule for Radon-Nikodym derivatives. We provide more detail to add insight. As in (9.31), Corollary 7.12 with $M_X = P_X = \pi_X = \mu_X$ yields (9.32) and (9.35) . The second equality is a result of rewriting $J(\pi_{XY}, \eta, s)$ by replacing the average distortion as an expectation involv-

ing the specially constructed joint distribution p_η:

$$
\begin{aligned}
&J(\pi_{XY}, \eta, s) \\
&= D(\pi_{XY} \| \pi_X \times \eta) + s\rho(\pi_{XY}) \\
&= \int d\pi_{XY}(x,y) \log \frac{d\pi_{XY}}{d(\pi_X \times \eta)}(x,y) - \int d\pi_{XY}(x,y) \log e^{-s\rho(x,y)} \\
&= \int d\pi_{XY}(x,y) \log \frac{d\pi_{XY}}{d(\pi_X \times \eta)}(x,y) \\
&\quad - \int d\pi_{XY}(x,y) \log \left(\gamma_{\eta,s}(x) e^{-s\rho(x,y)} \right) + \int d\pi_{XY}(x,y) \log \gamma_{\eta,s}(x) \\
&= \int d\pi_{XY}(x,y) \log \frac{d\pi_{XY}}{d(\mu_X \times \eta)}(x,y) \\
&\quad - \int d\pi_{XY}(x,y) \log \frac{dp_\eta}{d(\mu_X \times \eta)}(x,y) + \int d\mu_X(x) \log \gamma_{\eta,s}(x) \\
&= \int d\pi_{XY}(x,y) \log \left[\left(\frac{d\pi_{XY}}{d(\mu_X \times \eta)}(x,y) \right) / \left(\frac{dp_\eta}{d(\mu_X \times \eta)}(x,y) \right) \right] \\
&\quad + \int d\mu_X(x) \log \gamma_{\eta,s}(x) \\
&= \int d\pi_{XY}(x,y) \log \frac{d\pi_{XY}}{dp_\eta}(x,y) + \int d\mu_X(x) \log \gamma_{\eta,s}(x) \\
&= D(\pi_{XY} \| p_\eta) + \int d\mu_X(x) \log \gamma_{\eta,s}(x),
\end{aligned}
$$

which shows explicitly the Radon-Nikodym derivative chain rule application. We have that

$$
\begin{aligned}
&\int d\mu_X(x) \log \gamma_{\eta,s}(x) \\
&= \int dp_\eta(x,y) \log \gamma_{\eta,s}(x) = \int dp_\eta(x,y) \log \left(\gamma_{\eta,s}(x) e^{-s\rho(x,y)} e^{s\rho(x,y)} \right) \\
&= \int dp_\eta(x,y) \log \frac{dp_\eta}{d(\mu_X \times \eta)} + s \int dp_\eta(x,y) \rho(x,y) \\
&= D(p_\eta \| p_X \times \eta) + s\rho(p_\eta) = J(p_\eta, \eta, s),
\end{aligned}
$$

which completes the proof. □

The representations of the lemma imply immediate lower bounds to $J(\pi_{XY}, \eta, s)$ with obvious conditions for equality, as summarized in the following corollary.

Corollary 9.3.

$$J(\pi_{XY}, \eta, s) \geq J(\pi_{XY}, \pi_Y, s) \text{ with equality if } \eta = \pi_Y \tag{9.35}$$

$$J(\pi_{XY}, \eta, s) \geq \int d\mu_X(x) \log \gamma_{\eta,s}(x) \tag{9.36}$$

$$= J(p_\eta, \eta, s) \text{ with equality if } \pi_{XY} = p_\eta \tag{9.37}$$

$$F(s) = \inf_{\eta, \pi: \pi_X = \mu_X} J(\pi, \eta, s) \tag{9.38}$$

$$= \inf_\eta \int d\mu_X(x) \log \gamma_{\eta,s}(x)$$

$$= \inf_\eta \int d\mu_X(x) \log \frac{1}{\int d\eta(y) e^{-\rho(x,y)}}. \tag{9.39}$$

Proof. The inequalities and conditions for equality (9.35-9.37) follow directly from the lemma and the divergence inequality. Eq. (9.38) follows from the definition of $F(s)$ since

$$\inf_{\pi: \pi_X = \mu_X, \eta} J(\pi, \eta, s) = \inf_{\pi: \pi_X = \mu_X, \eta} [D(\pi \| \mu_X \times \eta) + s\rho(\pi)]$$

$$= \inf_{\pi: \pi_X = \mu_X} [D(\pi \| \pi_X \times \pi_Y) + s\rho(\pi)] = F(s)$$

and (9.39) follows since if we choose a reproduction distribution η within ϵ of the infimum, then using p_η yields $J(p_\eta, \eta, s)$ within ϵ of the infimum, and hence $F(s)$ can be no farther than ϵ from the infimum. Since ϵ is arbitrary, $F(s)$ must equal the infimum.

□

The corollary suggests a numerical algorithm for evaluating the rate distortion function. Given the input distribution μ_X, pick some reproduction distribution $\eta^{(0)}$. This $\eta^{(0)}$ together with μ_X implies a joint distribution $p^{(0)} = p_{\eta^{(0)}}$ with input marginal μ_X resulting in $J(p_{\eta^{(0)}}, \eta^{(0)}, s)$. Replace $\eta^{(0)}$ by $\eta^{(1)} = p_Y^{(0)}$, which yields $J(p_{\eta^{(0)}}, \eta^{(1)}, s) \leq J(p_{\eta^{(0)}}, \eta^{(0)}, s)$, that is, J can not increase. Then use the new reproduction marginal $\eta^{(1)}$ to form a new joint distribution $p_{(1)}$, which results in $J(p_{(1)}, \eta^{(1)}, s) \leq J(p_{\eta^{(0)}}, \eta^{(1)}, s)$. Continue in this matter, alternatively picking the best joint distribution for the reproduction and vice versa. Since J is monotonically nonincreasing and nonnegative, this is a descent algorithm and hence it must converge. This is the idea behind Blahut's algorithm [18] for computing the rate-distortion function. Blahut discretizes the problem by quantizing the input and output spaces to make the algorithm amenable to numerical solution. As discused by Rose [158], the algorithm can be sensitive to the nature of the discretization. In particular, a fixed quantization of source and reproduction can yield a suboptimal support for the reproduction distribution

The corollary shows that $F(s)$ can be stated as a optimization over the reproduction distribution as in (9.39). If an optimal reproduction

distribution η^* exists, then from Corollary 9.3 it must be true that the optimal joint distribution is $\pi^*_{X,Y}$ with

$$\pi^*_{X,Y} = p_{\eta^*} \tag{9.40}$$

$$\eta^* = \pi^*_Y \tag{9.41}$$

since otherwise either $D(\pi_{XY}\|p_{\eta^*})$ or $D(\pi_Y\|\eta^*)$ would be nonzero and hence $J(\pi_{X,Y},\eta,s)$ could be further decreased towards its infimum by substituting the appropriate joint or reproduction distribution. If the optimal reproduction distribution exists, it is called the *Shannon optimal reproduction distribution*. If these optimal distributions exist, then together they induce a regular conditional probability measure $P(X \in F \mid Y = y)$ given by

$$P(X \in F \mid Y = y) = \int_F d\mu(x)\gamma_{\eta^*,s}(x)e^{-s\rho(x,y)},$$

so that $\gamma_{\eta^*,s}(x)e^{-s\rho(x,y)}$ has the interpretation of being the *backward test channel* of the input given the output.

The following theorem summarizes the results developed in this section. It comprises a combination of Lemma 1.2, corollary to Lemma 1.3, and equations (1.11) and (1.15) in Csiszár [25].

Theorem 9.2. *If $R(D) < \infty$, then*

$$R(D) = \max_{s\geq 0}(F(s) - sD) \tag{9.42}$$

$$F(s) = \inf_{\pi\in\mathcal{P}(\mu_X)}(I(\pi) + sd(\pi)) \tag{9.43}$$

$$= \inf_{\mu_Y}\int d\mu_X(x)\log\frac{1}{\int d\mu_Y(y)e^{-sd(x,y)}} \tag{9.44}$$

where the final line defines $F(s)$ as an infimum over all distributions on $\hat{A}$. There exists a value s such that the straight line of slope $-s$ is tangent to the rate-distortion curve at $(R(D),D)$, in which case s is said to be associated with D. *If π achieves a minimum in (9.43), then $D = d(\pi), R(D) = I(\pi)$.*

Thus for a given D there is a value of s associated with D, and for this value the evaluation of the rate-distortion curve can be accomplished by an optimization over all distributions μ_Y on the reproduction alphabet. If a minimizing π exists, then the resulting marginal distribution for μ_Y is called a *Shannon optimal reproduction distribution*. In general this distribution need not be unique.

Csiszár [25] goes on to develop necessary and sufficient conditions for solutions to the optimizations defining $F(s)$ and $R(D)$, but the above results suffice for our purpose of demonstrated the role of the divergence

inequality in the optimization, and sketching the basic ideas underlying numerical algorithms for computing the rate-distortion function and the properties of optimal distributions in the Shannon sense. Conditions for the existence of solutions and for their uniqueness are also developed in [25]. We here state without proof one such result which will be useful in the discussion of optimality properties of source codes. The result shows that under the assumptions of a distortion measure that is a power of a metric derived from a norm, there exists a π achieving the minimum of (9.2) and hence also a Shannon optimal reproduction distribution. Both the lemma and the subsequent corollary are implied by the proof of Csiszár's Theorem 2.2 and the extension of the reproduction space from compact metric to Euclidean spaces discussed at the bottom of p. 66 of [25]. In the corollary, the roles of distortion and mutual information are interchanged to obtain the distortion-rate version of the result.

Lemma 9.9. *Given a random vector X with an alphabet A which is a finite-dimensional Euclidean space with norm $\|x\|$, a reproduction alphabet $\hat{A} = A$, and a distortion measure $d(x, y) = \|x - y\|^r$, $r > 0$, then there exists a distribution π on $A \times A$ achieving the the minimum of (9.2). Hence a Shannon N-dimensional optimal reproduction distribution exists for the Nth order rate-distortion function.*

Corollary 9.4. *Given the assumptions of the lemma, suppose that $\pi^{(n)}$, $n = 1, 2, \ldots$ is sequence of distributions on $A \times \hat{A}$ with marginals μ_X and $\mu_{Y^{(n)}}$ for which for $n = 1, 2, \ldots$*

$$I(\pi^{(n)}) = I(X, Y^{(n)}) \le R, \tag{9.45}$$

$$\lim_{n \to \infty} E[d(X, Y^{(n)})] = D_X(R). \tag{9.46}$$

Then $\mu_{Y^{(n)}}$ has a subsequence that converges weakly to a Shannon optimal reproduction distribution. If the Shannon distribution is unique, then $\mu_{Y^{(n)}}$ converges weakly to it.

The result is proved by showing that the stated conditions imply that any sequence of distributions $\pi^{(n)}$ has a weakly converging subsequence and that the limiting distribution inherits the properties of the individual $\pi^{(n)}$. If the Shannon optimal distribution is unique, then we can assume that $\mu_{Y_0}^{(n)}$ converges weakly to it.

Note that if there is a unique Shannon optimal reproduction distribution, then any sequence of $\pi^{(n)}$ for which (9.45–9.46) hold must converge weakly to the optimal distribution.

Support of Shannon Optimal Distributions

We close this chapter with a discussion of some of the interesting aspects of the Shannon optimal distribution. It is rare that an analytical formula is known for the distribution, one of the notable exceptions being a Gaussian IID source with variance σ^2. In this case the N-dimensional Shannon optimal reproduction distributions are known to be the product of N Gaussian distributions with variance $\sigma^2 - D$. In particular, the reproduction distribution is continuous. This turns out to be an exception, a fact which has had an effect on the evaluation of rate-distortion functions in the past. Some of the history and issues are discussed here. The discussion follows that of [117].

The basic ideas behind Blahut's algorithm were described in Section 9.5. The algorithm works quite well for discrete sources, but historically it has been applied to continuous sources in a way that often provided incorrect or misleading results. In particular, the standard approach in the literature was to first quantize the source and reproduction alphabet and then run the algorithm on the resulting discrete source. As pointed out by S. Fix [43], the reproduction alphabet chosen in this way was arbitrary and unchangeable by the algorithm itself. Fix proved that in the case of the squared error distortion measure, it is often the case that the optimal reproduction alphabet has finite support, that is, is concentrated on a specific finite set. If the initial quantization prior to the Blahut algorithm does not take this into account, the subsequent optimization can yield a poor solution to the original problem. This is not an uncommon problem since, as Fix showed, the optimal reproduction algorithm has finite support whenever a lower bound to the RDF due to Shannon [163], the Shannon lower bound, does *not* hold with equality. This occurs often for common sources and distortion measures. In fact the IID Gaussian source with a squared error distortion and IID discrete sources with a Hamming distortion are the only commonly encountered cases where the Shannon lower bound *does* hold with equality. A classic example of the problem is with the simple uniform IID source and a squared error distortion. Here the optimum reproduction alphabet is not only finite, but it can be small — only three letters for a rate of 1 bit per symbol. Inaccurate values for the RDF for this case based on Blahut's algorithm have been reported in the literature. A similar problem arises with discrete sources if one is given the option finding an optimal reconstruction alphabet instead of assuming that it is the same as the input alphabet. Early work on this problem was considered by T. Benjamin [9]. In such cases the Blahut algorithm only adjusts the probabilities assigned to the assumed reproduction alphabet, it does not seek an optimum alphabet. As pointed out by Fix, the general optimization problem can be formulated, but it is a nonlinear optimization and no one approach is clearly best. K. Rose [158] extended Fix's result showing finite support

of the optimal reproduction distribution when the Shannon lower bound is not met. He developed a deterministic annealing algorithm with supporting arguments and impressive experimental evidence showing that the algorithm found the optimal reproduction alphabets and the best existing estimates of the RDFs for several examples of IID sources.

Regrettably Fix's fascinating work was never formally published outside of his dissertation, and the mathematical details are long and complicated. His arguments are based on Csiszár's [25] careful development, which is partially developed in Section 9.5. Csiszár considered in depth the issues of the existence of solutions and the asymptotics and his paper is the definitive reference for the most general known results of this variety. The results of Fix and Rose, however, add an important aspect to the problem by pointing out that the choice of the support of the reproduction distribution must be considered if accurate results are to be obtained. This is important not only for the evaluation of the rate-distortion functions, but to the characterization of approximately optimal codes, as will be considered in Chapter 13.

Chapter 10
Relative Entropy Rates

Abstract Many of the basic properties of relative entropy are extended to sequences of random variables and to processes. Several limiting properties of entropy rates are proved and a mean ergodic theorem for relative entropy densities is given. The principal ergodic theorems for relative entropy and information densities in the general case are given in the next chapter.

10.1 Relative Entropy Densities and Rates

Suppose that p and m are two AMS distributions for a random process $\{X_n\}$ with a standard alphabet A. For convenience we assume that the random variables $\{X_n\}$ are coordinate functions of an underlying measurable space $(\Omega, \mathcal{B})$ where Ω is a one-sided or two-sided sequence space and $\mathcal{B}$ is the corresponding σ-field. Thus $x \in \Omega$ has the form $x = \{x_i\}$, where the index i runs from 0 to ∞ for a one-sided process and from $-\infty$ to $+\infty$ for a two-sided process. The random variables and vectors of principal interest are $X_n(x) = x_n$, $X^n(x) = x^n = (x_0, \cdots, x_{n-1})$, and $X_l^k(x) = (x_l, \cdots, x_{l+k-1})$. The process distributions p and m are both probability measures on the measurable space $(\Omega, \mathcal{B})$.

For $n = 1, 2, \ldots$ let M_{X^n} and P_{X^n} be the vector distributions induced by p and m. We assume throughout this section that $M_{X^n} \gg P_{X^n}$ and hence that the Radon-Nikodym derivatives $f_{X^n} = dP_{X^n}/dM_{X^n}$ and the entropy densities $h_{X^n} = \ln f_{X^n}$ are well defined for all $n = 1, 2, \ldots$ Strictly speaking, for each n the random variable f_{X^n} is defined on the measurable space $(A^n, \mathcal{B}_{A^n})$ and hence f_{X^n} is defined on a different space for each n. When considering convergence of relative entropy densities, it is necessary to consider a sequence of random variables defined on a common measurable space, and hence two notational modifications are introduced: The random variables $f_{X^n}(X^n) : \Omega \to [0, \infty)$ are defined by

R.M. Gray, *Entropy and Information Theory*, DOI 10.1007/978-1-4419-7970-4_10,

$$f_{X^n}(X^n)(x) \equiv f_{X^n}(X^n(x)) = f_{X^n}(x^n)$$

for $n = 1, 2, \ldots$. Similarly the entropy densities can be defined on the common space $(\Omega, \mathcal{B})$ by

$$h_{X^n}(X^n) = \ln f_{X^n}(X^n).$$

The reader is warned of the potentially confusing dual use of X^n in this notation: the subscript is the name of the random variable X^n and the argument is the random variable X^n itself. To simplify notation somewhat, we will often abbreviate the previous (unconditional) densities to

$$f_n = f_{X^n}(X^n); \; h_n = h_{X^n}(X^n).$$

For $n = 1, 2, \ldots$ define the relative entropy by

$$H_{p\|m}(X^n) = D(P_{X^n}\|M_{X^n}) = E_{P_{X^n}} h_{X^n} = E_p h_{X^n}(X^n).$$

Define the *relative entropy rate* by

$$\overline{H}_{p\|m}(X) = \limsup_{n\to\infty} \frac{1}{n} H_{p\|m}(X^n).$$

Analogous to Dobrushin's definition of information rate, we also define

$$H^*_{p\|m}(X) = \sup_q \overline{H}_{p\|m}(q(X)),$$

where the supremum is over all scalar quantizers q.

Define as in Chapter 7 the conditional densities

$$f_{X_n|X^n} = \frac{f_{X^{n+1}}}{f_{X^n}} = \frac{dP_{X^{n+1}}/dM_{X^{n+1}}}{dP_{X^n}/dM_{X^n}} = \frac{dP_{X_n|X^n}}{dM_{X_n|X^n}} \tag{10.1}$$

provided $f_{X^n} \neq 0$ and $f_{X_n|X^n} = 1$ otherwise. As for unconditional densities we change the notation when we wish to emphasize that the densities can all be defined on a common underlying sequence space. For example, we follow the notation for ordinary conditional probability density functions and define the random variables

$$f_{X_n|X^n}(X_n|X^n) = \frac{f_{X^{n+1}}(X^{n+1})}{f_{X^n}(X^n)}$$

and

$$h_{X_n|X^n}(X_n|X^n) = \ln f_{X_n|X^n}(X_n|X^n)$$

on $(\Omega, \mathcal{B})$. These densities will not have a simple abbreviation as do the unconditional densities.

Define the conditional relative entropy

$$H_{p\|m}(X_n|X^n) = E_{P_{X^n}}(\ln f_{X_n|X^n}) = \int dp \ln f_{X_n|X^n}(X_n|X^n). \tag{10.2}$$

All of the above definitions are immediate applications of definitions of Chapter 7 to the random variables X_n and X^n. The difference is that these are now defined for all samples of a random process, that is, for all $n = 1, 2, \ldots$. The focus of this chapter is the interrelations of these entropy measures and on some of their limiting properties for large n.

For convenience define

$$D_n = H_{p\|m}(X_n|X^n);\ n = 1, 2, \ldots,$$

and $D_0 = H_{p\|m}(X_0)$. From Theorem 7.2 this quantity is nonnegative and

$$D_n + D(P_{X^n}\|M_{X^n}) = D(P_{X^{n+1}}\|M_{X^{n+1}}).$$

If $D(P_{X^n}\|M_{X^n}) < \infty$, then also

$$D_n = D(P_{X^{n+1}}\|M_{X^{n+1}}) - D(P_{X^n}\|M_{X^n}).$$

We can write D_n as a single divergence if we define as in Theorem 7.2 the distribution $S_{X^{n+1}}$ by

$$S_{X^{n+1}}(F \times G) = \int_F M_{X_n|X^n}(F|x^n)\, dP_{X^n}(x^n);\ F \in \mathcal{B}_A;\ G \in \mathcal{B}_{A^n}. \tag{10.3}$$

Recall that $S_{X^{n+1}}$ combines the distribution P_{X^n} on X^n with the conditional distribution $M_{X_n|X^n}$ giving the conditional probability under M for X_n given X^n. We shall abbreviate this construction by

$$S_{X^{n+1}} = \overline{M_{X_n|X^n} P_{X^n}}. \tag{10.4}$$

Then

$$D_n = D(P_{X^{n+1}}\|S_{X^{n+1}}). \tag{10.5}$$

Note that $S_{X^{n+1}}$ is not in general a consistent family of measures in the sense of the Kolmogorov extension theorem since its form changes with n, the first n samples being chosen according to p and the final sample being chosen using the conditional distribution induced by m given the first n samples. Thus, in particular, we cannot infer that there is a process distribution s which has $S_{X^n};\ n = 1, 2, \ldots$ as its vector distributions.

We immediately have a chain rule for densities

$$f_{X^n} = \prod_{i=0}^{n-1} f_{X_i|X^i} \tag{10.6}$$

and a corresponding chain rule for conditional relative entropies similar to that for ordinary entropies:

$$D(P_{X^n}\|M_{X^n}) = H_{p\|m}(X^n) = \sum_{i=0}^{n-1} H_{p\|m}(X_i|X^i) = \sum_{i=0}^{n-1} D_i. \tag{10.7}$$

10.2 Markov Dominating Measures

The evaluation of relative entropy simplifies for certain special cases and reduces to a mutual information when the dominating measure is a Markov approximation of the dominated measure. The following lemma is an extension to sequences of the results of Corollary 7.13 and Lemma 7.17.

Theorem 10.1. *Suppose that p is a process distribution for a standard alphabet random process $\{X_n\}$ with induced vector distributions P_{X^n}; $n = 1, 2, \ldots$. Suppose also that there exists a process distribution m with induced vector distributions M_{X^n} such that*

(a)under m $\{X_n\}$ is a k-step Markov source, that is, for all $n \geq k$, $X^{n-k} \to X_{n-k}^k \to X_n$ is a Markov chain or, equivalently,

$$M_{X_n|X^n} = M_{X_n|X_{n-k}^k},$$

and

(b)$M_{X^n} \gg P_{X^n}$, $n = 1, 2, \ldots$ so that the densities

$$f_{X^n} = \frac{dP_{X^n}}{dM_{X^n}}$$

are well defined.

Suppose also that $p^{(k)}$ is the k-step Markov approximation to p, that is, the source with induced vector distributions $P_{X^n}^{(k)}$ such that

$$P_{X^k}^{(k)} = P_{X^k}$$

and for all $n \geq k$

$$P_{X_n|X^n}^{(k)} = P_{X_n|X_{n-k}^k};$$

that is, $p^{(k)}$ is a k-step Markov process having the same initial distribution and the same kth order conditional probabilities as p. Then for all $n \geq k$

$$M_{X^n} \gg P_{X^n}^{(k)} \gg P_{X^n} \tag{10.8}$$

and

$$\frac{dP_{X^n}^{(k)}}{dM_{X^n}} = f_{X^n}^{(k)} \equiv f_{X^k} \prod_{l=k}^{n-1} f_{X_l|X_{l-k}^k}, \tag{10.9}$$

$$\frac{dP_{X^n}}{dP_{X^n}^{(k)}} = \frac{f_{X^n}}{f_{X^n}^{(k)}}. \tag{10.10}$$

Furthermore

$$h_{X_n|X^n} = h_{X_n|X_{n-k}^k} + i_{X_n;X^{n-k}|X_{n-k}^k} \tag{10.11}$$

and hence

$$\begin{aligned} D_n &= H_{p\|m}(X_n|X^n) \\ &= I_p(X_n;X^{n-k}|X_{n-k}^k) + H_{p\|m}(X_n|X_{n-k}^k). \end{aligned}$$

Thus

$$h_{X^n} = h_{X^k} + \sum_{l=k}^{n-1} \left(h_{X_l|X_{l-k}^k} + i_{X_l;X^{l-k}|X_{l-k}^k} \right) \tag{10.12}$$

and hence

$$D(P_{X^n}\|M_{X^n}) = H_{p\|m}(X^k) + \sum_{l=k}^{n-1} (I_p(X_l;X^{l-k}|X_{l-k}^k) + H_{p\|m}(X_l|X_{l-k}^k)). \tag{10.13}$$

If $m = p^{(k)}$, then for all $n \geq k$ we have that $h_{X_n|X_{n-k}^k} = 0$ and hence

$$H_{p\|p^{(k)}}(X_n|X_{n-k}^k) = 0 \tag{10.14}$$

and

$$D_n = I_p(X_n;X^{n-k}|X_{n-k}^k), \tag{10.15}$$

and hence

$$D(P_{X^n}\|P_{X^n}^{(k)}) = \sum_{l=k}^{n-1} I_p(X_l;X^{l-k}|X_{l-k}^k). \tag{10.16}$$

Proof: If $n = k + 1$, then the results follow from Corollary 7.9 and Lemma 7.17 with $X = X_n$, $Z = X^k$, and $Y = X_k$. Now proceed by induction and assume that the results hold for n. Consider the distribution $Q_{X^{(n+1)}}$ specified by $Q_{X^n} = P_{X^n}$ and $Q_{X_n|X^n} = P_{X_n|X_{n-k}^k}$. In other words,

$$Q_{X^{n+1}} = \overline{P_{X_n|X_{n-k}^k} P_{X^n}}$$

Application of Corollary 7.7 withright-hand $Z = X^{n-k}$, $Y = X_{n-k}^k$, and $X = X_n$ implies that $M_{X^{n+1}} \gg Q_{X^{n+1}} \gg P_{X^{n+1}}$ and that

$$\frac{dP_{X^{n+1}}}{dQ_{X^{n+1}}} = \frac{f_{X_n|X^n}}{f_{X_n|X_{n-k}^k}}.$$

This means that we can write

$$\begin{aligned} P_{X^{n+1}}(F) &= \int_F \frac{dP_{X^{n+1}}}{dQ_{X^{n+1}}}\, dQ_{X^{n+1}} = \int_F \frac{dP_{X^{n+1}}}{dQ_{X^{n+1}}}\, dQ_{X_n|X^n}\, dQ_{X^n} \\ &= \int_F \frac{dP_{X^{n+1}}}{dQ_{X^{n+1}}}\, dP_{X_n|X_{n-k}^k}\, dP_{X^n}. \end{aligned}$$

From the induction hypothesis we can express this as

$$\begin{aligned} P_{X^{n+1}}(F) &= \int_F \frac{dP_{X^{n+1}}}{dQ_{X^{n+1}}} \frac{dP_{X^n}}{dP_{X^n}^{(k)}}\, dP_{X_n|X_{n-k}^k}\, dP_{X^n}^{(k)} \\ &= \int_F \frac{dP_{X^{n+1}}}{dQ_{X^{n+1}}} \frac{dP_{X^n}}{dP_{X^n}^{(k)}} dP_{X^{n+1}}^{(k)}, \end{aligned}$$

proving that $P_{X^{n+1}}^{(k)} \gg P_{X^{n+1}}$ and that

$$\frac{dP_{X^{n+1}}}{dP_{X^{n+1}}^{(k)}} = \frac{dP_{X^{n+1}}}{dQ_{X^{n+1}}} \frac{dP_{X^n}}{dP_{X^n}^{(k)}} = \frac{f_{X_n|X^n}}{f_{X_n|X_{n-k}^k}} \frac{dP_{X^n}}{dP_{X^n}^{(k)}}.$$

This proves the right-hand part of (10.9) and (10.10).

Next define the distribution $\hat{P}_{X^n}$ by

$$\hat{P}_{X^n}(F) = \int_F f_{X^n}^{(k)}\, dM_{X^n},$$

where $f_{X^n}^{(k)}$ is defined in (10.9). Proving that $\hat{P}_{X^n} = P_{X^n}^{(k)}$ will prove both the left hand relation of (10.8) and (10.9). Clearly

$$\frac{\hat{d}P_{X^n}}{dM_{X^n}} = f_{X^n}^{(k)}$$

and from the definition of $f^{(k)}$ and conditional densities

$$f_{X_n|X^n}^{(k)} = f_{X_n|X_{n-k}^k}^{(k)}. \tag{10.17}$$

From Corollary 7.7 it follows that $X^{n-k} \to X_{n-k}^k \to X_n$ is a Markov chain. Since this is true for any $n \geq k$, $\hat{P}_{X^n}$ is the distribution of a k-step Markov process. By construction we also have that

$$f_{X_n|X_{n-k}^k}^{(k)} = f_{X_n|X_{n-k}^k} \tag{10.18}$$

and hence from Theorem 7.2

$$P^{(k)}_{X_n|X^k_{n-k}} = P_{X_n|X^k_{n-k}}.$$

Since also $f^{(k)}_{X^k} = f_{X^k}$, $\hat{P}_{X^n} = P^{(k)}_{X^n}$ as claimed. This completes the proof of (10.8)–(10.10). Eq. (10.11) follows since

$$f_{X_n|X^n} = f_{X_n|X^k_{n-k}} \times \frac{f_{X_n|X^n}}{f_{X_n|X^k_{n-k}}}.$$

Eq. (10.12) then follows by taking expectations. Eq. (10.12) follows from (10.11) and

$$f_{X^n} = f_{X^k} \prod_{l=k}^{n-1} f_{X_l|X^l},$$

whence (10.13) follows by taking expectations. If $m = p^{(k)}$, then the claims follow from (7.23)–(7.24). □

Corollary 10.1. *Given a stationary source p, suppose that for some K there exists a K-step Markov source m with distributions $M_{X^n} \gg P_{X^n}$, $n = 1, 2, \ldots$. Then for all $k \geq K$ (10.8)–(10.10) hold.*

Proof: If m is a K-step Markov source with the property $M_{X^n} \gg P_{X^n}$, $n = 1, 2, \ldots$, then it is also a k-step Markov source with this property for all $k \geq K$. The corollary then follows from the theorem. □

Comment: The corollary implies that if *any* K-step Markov source dominates p on its finite dimensional distributions, then for *all* $k \geq K$ the k-step Markov approximations $p^{(k)}$ also dominate p on its finite dimensional distributions.

The following variational corollary follows from Theorem 10.1.

Corollary 10.2. *For a fixed k let Let $\mathcal{M}^k$ denote the set of all k-step Markov distributions. Then* $\inf_{M \in \mathcal{M}^k} D(P_{X^n} \| M)$ *is attained by $P^{(k)}$, and*

$$\inf_{M \in \mathcal{M}^k} D(P_{X^n} \| M) = D(P_{X^n} \| P^{(k)}_{X^n}) = \sum_{l=k}^{n-1} I_p(X_l; X^{l-k} | X^k_{l-k}).$$

Since the divergence can be thought of as a distance between probability distributions, the corollary justifies considering the k-step Markov process with the same kth order distributions as the k-step Markov *approximation* or *model* for the original process: It is the minimum divergence distribution meeting the k-step Markov requirement.

10.3 Stationary Processes

Several of the previous results simplify when the processes m and p are both stationary. We can consider the processes to be two-sided since given a stationary one-sided process, there is always a stationary two-sided process with the same probabilities on all positive time events. When both processes are stationary, the densities $f_{X_m^n}$ and f_{X^n} satisfy

$$f_{X_m^n} = \frac{dP_{X_m^n}}{dM_{X_m^n}} = f_{X^n}T^m = \frac{dP_{X^n}}{dM_{X^n}}T^m,$$

and have the same expectation for any integer m. Similarly the conditional densities $f_{X_n|X^n}$, $f_{X_k|X_{k-n}^n}$, and $f_{X_0|X_{-1},X_{-2},\cdots,X_{-n}}$ satisfy

$$f_{X_n|X^n} = f_{X_k|X_{k-n}^n}T^{n-k} = f_{X_0|X_{-1},X_{-2},\cdots,X_{-n}}T^n \tag{10.19}$$

for any k and have the same expectation. Thus

$$\frac{1}{n}H_{p\|m}(X^n) = \frac{1}{n}\sum_{i=0}^{n-1} H_{p\|m}(X_0|X_{-1},\cdots,X_{-i}). \tag{10.20}$$

Using the construction of Theorem 7.2 we have also that

$$\begin{aligned} D_i &= H_{p\|m}(X_i|X^i) = H_{p\|m}(X_0|X_{-1},\cdots,X_-i) \\ &= D(P_{X_0,X_{-1},\cdots,X_{-i}}\|S_{X_0,X_{-1},\cdots,X_{-i}}), \end{aligned}$$

where now

$$S_{X_0,X_{-1},\cdots,X_{-i}} = \overline{M_{X_0|X_{-1},\cdots,X_{-i}}P_{X_{-1},\cdots,X_{-i}}}; \tag{10.21}$$

that is,

$$S_{X_0,X_{-1},\cdots,X_{-i}}(F\times G) = \int_F M_{X_0|X_{-1},\cdots,X_{-i}}(F|x^i)\,dP_{X_{-1},\cdots,X_{-i}}(x^i); F\in\mathcal{B}_A; G\in\mathcal{B}_{A^i}.$$

As before the S_{X^n} distributions are not in general consistent. For example, they can yield differing marginal distributions S_{X_0}. As we saw in the finite case, general conclusions about the behavior of the limiting conditional relative entropies cannot be drawn for arbitrary reference measures. If, however, we assume as in the finite case that the reference measures are Markov, then we can proceed.

Suppose now that under m the process is a k-step Markov process. Then for any $n \ge k$ $(X_{-n},\cdots,X_{-k-2},X_{-k-1}) \to X_{-k}^k \to X_0$ is a Markov chain under m and Lemma 7.17 implies that

$$H_{p\|m}(X_0|X_{-1},\cdots,X_{-n}) = H_{p\|m}(X_k|X^k) + I_p(X_k;(X_{-1},\cdots,X_{-n})|X^k) \quad (10.22)$$

and hence from (10.20)

$$\tilde{H}_{p\|m}(X) = H_{p\|m}(X_k|X^k) + I_p(X_k;X^-|X^k). \quad (10.23)$$

We also have, however, that $X^- \to X^k \to X_k$ is a Markov chain under m and hence a second application of Lemma 7.17 implies that

$$H_{p\|m}(X_0|X^-) = H_{p\|m}(X_k|X^k) + I_p(X_k;X^-|X^k). \quad (10.24)$$

Putting these facts together and using (10.2) yields the following lemma.

Lemma 10.1. *Let $\{X_n\}$ be a two-sided process with a standard alphabet and let p and m be stationary process distributions such that $M_{X^n} \gg P_{X^n}$ all n and m is kth order Markov. Then the relative entropy rate exists and*

$$\begin{aligned}\overline{H}_{p\|m}(X) &= \lim_{n\to\infty} \frac{1}{n} H_{p\|m}(X^n)\\ &= \lim_{n\to\infty} H_{p\|m}(X_0|X_{-1},\cdots,X_{-n})\\ &= H_{p\|m}(X_0|X^-)\\ &= H_{p\|m}(X_k|X^k) + I_p(X_k;X^-|X^k)\\ &= E_p[\ln f_{X_k|X^k}(X_k|X^k)] + I_p(X_k;X^-|X^k).\end{aligned}$$

Corollary 10.3. *Given the assumptions of Lemma 10.1,*

$$H_{p\|m}(X^N|X^-) = NH_{p\|m}(X_0|X^-).$$

Proof: From the chain rule for conditional relative entropy (equation (10.7),

$$H_{p\|m}(X^N|X^-) = \sum_{l=0}^{n-1} H_{p\|m}(X_l|X^l,X^-).$$

Stationarity implies that each term in the sum equals $H_{p\|m}(X_0|X^-)$, proving the corollary. □

The next corollary extends Corollary 10.1 to processes.

Corollary 10.4. *Given k and $n \ge k$, let $\mathcal{M}^k$ denote the class of all k-step stationary Markov process distributions. Then*

$$\inf_{m\in\mathcal{M}^k} \overline{H}_{p\|m}(X) = \overline{H}_{p\|p^{(k)}}(X) = I_p(X_k;X^-|X^k).$$

Proof: Follows from (10.22) and Theorem 10.1. □

This result gives an interpretation of the finite-gap information property (8.15): If a process has this property, then there exists a k-step Markov process which is only a finite "distance" from the given process in terms of limiting per-symbol divergence. If any such process has a finite distance, then the k-step Markov approximation also has a finite distance. Furthermore, we can apply Corollary 8.4 to obtain the generalization of the finite alphabet result of Theorem 3.4

.

Corollary 10.5. *Given a stationary process distribution p which satisfies the finite-gap information property,*

$$\inf_k \inf_{m \in \mathcal{M}^k} \overline{H}_{p\|m}(X) = \inf_k \overline{H}_{p\|p^{(k)}}(X) = \lim_{k\to\infty} \overline{H}_{p\|p^{(k)}}(X) = 0.$$

Lemma 10.1 also yields the following approximation lemma.

Corollary 10.6. *Given a process $\{X_n\}$ with standard alphabet A let p and m be stationary measures such that $P_{X^n} \ll M_{X^n}$ for all n and m is kth order Markov. Let q_k be an asymptotically accurate sequence of quantizers for A. Then*

$$\overline{H}_{p\|m}(X) = \lim_{k\to\infty} \overline{H}_{p\|m}(q_k(X)),$$

that is, the divergence rate can be approximated arbitrarily closely by that of a quantized version of the process. Thus, in particular,

$$\overline{H}_{p\|m}(X) = H^*_{p\|m}(X).$$

Proof: This follows from Corollary 7.3 by letting the generating σ-fields be $\mathcal{F}_n = \sigma(q_n(X_i);\ i = 0, -1, \ldots)$ and the representation of conditional relative entropy as an ordinary divergence. □

Another interesting property of relative entropy rates for stationary processes is that we can "reverse time" when computing the rate in the sense of the following lemma.

Lemma 10.2. *Let $\{X_n\}$, p, and m be as in Lemma 10.1. If either $\overline{H}_{p\|m}(X) < \infty$ or $H_{P\|M}(X_0|X^-) < \infty$, then*

$$H_{p\|m}(X_0|X_{-1}, \cdots, X_{-n}) = H_{p\|m}(X_0|X_1, \cdots, X_n)$$

and hence

$$H_{p\|m}(X_0|X_1, X_2, \cdots) = H_{p\|m}(X_0\|X_{-1}, X_{-2}, \cdots) = \overline{H}_{p\|m}(X) < \infty.$$

Proof: If $\overline{H}_{p\|m}(X)$ is finite, then so must be the terms $H_{p\|m}(X^n) = D(P_{X^n}\|M_{X^n})$ (since otherwise all such terms with larger n would also be infinite and hence $\overline{H}$ could not be finite). Thus from stationarity

$$\begin{aligned} H_{p\|m}(X_0|X_{-1},\cdots,X_{-n}) &= H_{p\|m}(X_n|X^n) \\ &= D(P_{X^{n+1}}\|M_{X^{n+1}}) - D(P_{X^n}\|M_{X^n}) \\ D(P_{X^{n+1}}\|M_{X^{n+1}}) - D(P_{X_1^n}\|M_{X_1^n}) &= H_{p\|m}(X_0|X_1,\cdots,X_n) \end{aligned}$$

from which the results follow. If on the other hand the conditional relative entropy is finite, the results then follow as in the proof of Lemma 10.1 using the fact that the joint relative entropies are arithmetic averages of the conditional relative entropies and that the conditional relative entropy is defined as the divergence between the P and S measures (Theorem 7.3). □

10.4 Mean Ergodic Theorems

In this section we state and prove some preliminary ergodic theorems for relative entropy densities analogous to those first developed for entropy densities in Chapter 4 and for information densities in Section 8.3. In particular, we show that an almost everywhere ergodic theorem for finite alphabet processes follows easily from the sample entropy ergodic theorem and that an approximation argument then yields an L^1 ergodic theorem for stationary sources. The results involve little new and closely parallel those for mutual information densities and therefore the details are skimpy. The results are given for completeness and because the L^1 results yield the byproduct that relative entropies are uniformly integrable, a fact which does not follow as easily for relative entropies as it did for entropies.

Finite Alphabets

Suppose that we now have two process distributions p and m for a random process $\{X_n\}$ with finite alphabet. Let P_{X^n} and M_{X^n} denote the induced nth order distributions and p_{X^n} and m_{X^n} the corresponding probability mass functions (pmf's). For example, $p_{X^n}(a^n) = P_{X^n}(\{x^n : x^n = a^n\}) = p(\{x : X^n(x) = a^n\})$. We assume that $P_{X^n} \ll M_{X^n}$. In this case the relative entropy density is given simply by

$$h_n(x) = h_{X^n}(X^n)(x) = \ln \frac{p_{X^n}(x^n)}{m_{X^n}(x^n)},$$

where $x^n = X^n(x)$.

The following lemma generalizes Theorem 4.1 from entropy densities to relative entropy densities for finite alphabet processes. Relative entropies are of more general interest than ordinary entropies because they generalize to continuous alphabets in a useful way while ordinary entropies do not.

Lemma 10.3. *Suppose that $\{X_n\}$ is a finite alphabet process and that p and m are two process distributions with $M_{X^n} \gg P_{X^n}$ for all n, where p is AMS with stationary mean $\bar{p}$, m is a kth order Markov source with stationary transitions, and $\{\bar{p}_x\}$ is the ergodic decomposition of the stationary mean of p. Assume also that $M_{X^n} \gg \bar{P}_{X^n}$ for all n. Then*

$$\lim_{n\to\infty} \frac{1}{n} h_n = h; \ p - a.e. \text{ and in } L^1(p),$$

where $h(x)$ is the invariant function defined by

$$\begin{aligned} h(x) &= -\overline{H}_{\bar{p}_x}(X) - E_{\bar{p}_x} \ln m(X_k|X^k) \\ &= \lim_{n\to\infty} \frac{1}{n} H_{\bar{p}_x \| m}(X^n) = \overline{H}_{\bar{p}_x \| m}(X), \end{aligned}$$

where

$$m(X_k|X^k)(x) \equiv \frac{m_{X^{k+1}}(x^{k+1})}{m_{X^k}(x^k)} = M_{X_k|X^k}(x_k|x^k).$$

Furthermore,

$$E_p h = \overline{H}_{p\|m}(X) = \lim_{n\to\infty} \frac{1}{n} H_{p\|m}(X^n), \tag{10.25}$$

that is, the relative entropy rate of an AMS process with respect to a Markov process with stationary transitions is given by the limit. Lastly,

$$\overline{H}_{p\|m}(X) = \overline{H}_{\bar{p}\|m}(X); \tag{10.26}$$

that is, the relative entropy rate of the AMS process with respect to m is the same as that of its stationary mean with respect to m.

Proof: We have that

$$\begin{aligned} \frac{1}{n} h(X^n) &= \frac{1}{n} \ln p(X^n) - \frac{1}{n} \ln m(X^k) + \frac{1}{n} \sum_{i=k}^{n-1} \ln m(X_i|X_{i-k}^k) \\ &= \frac{1}{n} \ln p(X^n) - \frac{1}{n} \ln m(X^k) - \frac{1}{n} \sum_{i=k}^{n-1} \ln m(X_k|X^k) T^{i-k}, \end{aligned}$$

where T is the shift transformation, $p(X^n)$ is an abbreviation for $P_{X^n}(X^n)$, and $m(X_k|X^k) = M_{X_k|X^k}(X_k|X^k)$. From Theorem 4.1 the first term converges to $-\overline{H}_{\bar{p}_x}(X)$ p-a.e. and in $L^1(p)$.

Since $M_{X^k} \gg P_{X^k}$, if $M_{X^k}(F) = 0$, then also $P_{X^k}(F) = 0$. Thus P_{X^k} and hence also p assign zero probability to the event that $M_{X^k}(X^k) = 0$. Thus with probability one under p, $\ln m(X^k)$ is finite and hence the second term in 10.27 converges to 0 p-a.e. as $n \to \infty$.

Define α as the minimum nonzero value of the conditional probability $m(x_k|x^k)$. Then with probability 1 under M_{X^n} and hence also under P_{X^n} we have that

$$\frac{1}{n}\sum_{i=k}^{n-1} \ln \frac{1}{m(X_i|X_{i-k}^k)} \le \ln \frac{1}{\alpha}$$

since otherwise the sequence X^n would have 0 probability under M_{X^n} and hence also under P_{X^n} and $0 \ln 0$ is considered to be 0. Thus the rightmost term of (10.27) is uniformly integrable with respect to p and hence from Theorem 1.6 this term converges to $E_{\bar{p}_x}(\ln m(X_k|X^k))$. This proves the leftmost equality of (10.25).

Let $\bar{p}_{X^n|\mathcal{X}}$ denote the distribution of X^n under the ergodic component $\bar{p}_x$. Since $M_{X^n} \gg \bar{P}_{X^n}$ and $\bar{P}_{X^n} = \int d\bar{p}(x)\bar{p}_{X^n|\mathcal{X}}$, if $M_{X^n}(F) = 0$, then $\bar{p}_{X^n|\mathcal{X}}(F) = 0$ p-a.e. Since the alphabet of X_n if finite, we therefore also have with probability one under $\bar{p}$ that $M_{X^n} \gg \bar{p}_{X^n|\mathcal{X}}$ and hence

$$H_{\bar{p}_x\|m}(X^n) = \sum_{a^n} \bar{p}_{X^n|\mathcal{X}}(a^n) \ln \frac{\bar{p}_{X^n|\mathcal{X}}(a^n)}{M_{X^n}(a^n)}$$

is well defined for $\bar{p}$-almost all x. This expectation can also be written as

$$\begin{aligned} H_{\bar{p}_x\|m}(X^n) &= -H_{\bar{p}_x}(X^n) - E_{\bar{p}_x}[\ln m(X^k) + \sum_{i=k}^{n-1} \ln m(X_k|X^k)T^{i-k}] \\ &= -H_{\bar{p}_x}(X^n) - E_{\bar{p}_x}[\ln m(X^k)] - (n-k)E_{\bar{p}_x}[\ln m(X_k|X^k)], \end{aligned}$$

where we have used the stationarity of the ergodic components. Dividing by n and taking the limit as $n \to \infty$, the middle term goes to zero as previously and the remaining limits prove the middle equality and hence the rightmost inequality in (10.25).

Equation (10.25) follows from (10.25) and $L^1(p)$ convergence, that is, since $n^{-1}h_n \to h$, we must also have that $E_p(n^{-1}h_n(X^n)) = n^{-1}H_{p\|m}(X^n)$ converges to $E_p h$. Since the former limit is $\overline{H}_{p\|m}(X)$, (10.25) follows. Since $\bar{p}_x$ is invariant (Theorem 1.5) and since expectations of invariant functions are the same under an AMS measure and its stationary mean (Lemma 6.3.1 of [55] or Lemma 7.5 of [58]), application of the previous results of the lemma to both p and $\bar{p}$ proves that

$$\overline{H}_{p\|m}(X) = \int dp(x)\overline{H}_{\bar{p}_x\|m}(X) = \int d\bar{p}(x)\overline{H}_{\bar{p}_x\|m}(X) = \overline{H}_{\bar{p}\|m}(X),$$

which proves (10.27) and completes the proof of the lemma. □

Corollary 10.7. *Given p and m as in the Lemma, then the relative entropy rate of p with respect to m has an ergodic decomposition, that is,*

$$\overline{H}_{p\|m}(X) = \int dp(x)\overline{H}_{\bar{p}_x\|m}(X).$$

Proof: This follows immediately from (10.25) and (10.25). □

Standard Alphabets

We now drop the finite alphabet assumption and suppose that $\{X_n\}$ is a standard alphabet process with process distributions p and m, where p is stationary, m is kth order Markov with stationary transitions, and $M_{X^n} \gg P_{X^n}$ are the induced vector distributions for $n = 1, 2, \ldots$. Define the densities f_n and entropy densities h_n as previously.

As an easy consequence of the development to this point, the ergodic decomposition for divergence rate of finite alphabet processes combined with the definition of H^* as a supremum over rates of quantized processes yields an extension of Corollary 8.2 to divergences. This yields other useful properties as summarized in the following corollary.

Corollary 10.8. *Given a standard alphabet process $\{X_n\}$ suppose that p and m are two process distributions such that p is AMS and m is kth order Markov with stationary transitions and $M_{X^n} \gg P_{X^n}$ are the induced vector distributions. Let $\bar{p}$ denote the stationary mean of p and let $\{\bar{p}_x\}$ denote the ergodic decomposition of the stationary mean $\bar{p}$. Then*

$$H^*_{p\|m}(X) = \int dp(x)H^*_{\bar{p}_x\|m}(X). \tag{10.27}$$

In addition,

$$H^*_{p\|m}(X) = H^*_{\bar{p}\|m}(X) = \overline{H}_{\bar{p}\|m}(X) = \overline{H}_{p\|m}(X); \tag{10.28}$$

that is, the two definitions of relative entropy rate yield the same values for AMS p and stationary transition Markov m and both rates are the same as the corresponding rates for the stationary mean. Thus relative entropy rate has an ergodic decomposition in the sense that

$$\overline{H}_{p\|m}(X) = \int dp(x)\overline{H}_{\bar{p}_x\|m}(X). \tag{10.29}$$

Comment: Note that the extra technical conditions of Theorem 8.3 for equality of the analogous mutual information rates $\bar{I}$ and I^* are not needed here. Note also that only the ergodic decomposition of the sta-

tionary mean $\bar{p}$ of the AMS measure p is considered and not that of the Markov source m.

Proof: The first statement follows as previously described from the finite alphabet result and the definition of H^*. The left-most and right-most equalities of (10.28) both follow from the previous lemma. The middle equality of (10.28) follows from Corollary 10.4. Eq. (10.29) then follows from (10.27) and (10.28). □

Theorem 10.2. *Given a standard alphabet process $\{X_n\}$ suppose that p and m are two process distributions such that p is AMS and m is kth order Markov with stationary transitions and $M_{X^n} \gg P_{X^n}$ are the induced vector distributions. Let $\{\bar{p}_x\}$ denote the ergodic decomposition of the stationary mean $\bar{p}$. If*

$$\lim_{n\to\infty} \frac{1}{n} H_{p\|m}(X^n) = \overline{H}_{p\|m}(X) < \infty,$$

then there is an invariant function h such that $n^{-1}h_n \to h$ in $L^1(p)$ as $n \to \infty$. In fact,

$$h(x) = \overline{H}_{\bar{p}_x\|m}(X),$$

the relative entropy rate of the ergodic component $\bar{p}_x$ with respect to m. Thus, in particular, under the stated conditions the relative entropy densities h_n are uniformly integrable with respect to p.

Proof: The proof exactly parallels that of Theorem 8.1, the mean ergodic theorem for information densities, with the relative entropy densities replacing the mutual information densities. The density is approximated by that of a quantized version and the integral bounded above using the triangle inequality. One term goes to zero from the finite alphabet case. Since $\overline{H} = H^*$ (Corollary 10.8) the remaining terms go to zero because the relative entropy rate can be approximated arbitrarily closely by that of a quantized process. □

It should be emphasized that although Theorem 10.2 and Theorem 8.1 are similar in appearance, neither result directly implies the other. It is true that mutual information can be considered as a special case of relative entropy, but given a pair process $\{X_n, Y_n\}$ we cannot in general find a kth order Markov distribution m for which the mutual information rate $\bar{I}(X;Y)$ equals a relative entropy rate $\overline{H}_{p\|m}$. We will later consider conditions under which convergence of relative entropy densities does imply convergence of information densities.

Chapter 11
Ergodic Theorems for Densities

Abstract This chapter is devoted to developing ergodic theorems first for relative entropy densities and then information densities for the general case of AMS processes with standard alphabets. The general results were first developed by Barron using the martingale convergence theorem and a new martingale inequality. The similar results of Algoet and Cover can be proved without direct recourse to martingale theory. They infer the result for the stationary Markov approximation and for the infinite order approximation from the ordinary ergodic theorem. They then demonstrate that the growth rate of the true density is asymptotically sandwiched between that for the kth order Markov approximation and the infinite order approximation and that no gap is left between these asymptotic upper and lower bounds in the limit as $k \to \infty$. They use martingale theory to show that the values between which the limiting density is sandwiched are arbitrarily close to each other, but in this chapter it is shown that martingale theory is not needed and this property follows from the results of Chapter 8.

11.1 Stationary Ergodic Sources

Theorem 11.1. *Given a standard alphabet process $\{X_n\}$, suppose that p and m are two process distributions such that p is stationary ergodic and m is a K-step Markov source with stationary transition probabilities. Let $M_{X^n} \gg P_{X^n}$ be the vector distributions induced by p and m. As before let*

$$h_n = \ln f_{X^n}(X^n) = \ln \frac{dP_{X^n}}{dM_{X^n}}(X^n).$$

Then with probability one under p

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$$\lim_{n\to\infty} \frac{1}{n} h_n = \overline{H}_{p\|m}(X).$$

Proof: Let $p^{(k)}$ denote the k-step Markov approximation of p as defined in Theorem 10.1, that is, $p^{(k)}$ has the same kth order conditional probabilities and k-dimensional initial distribution. From Corollary 10.1, if $k \geq K$, then (10.8)–(10.10) hold. Consider the expectation

$$E_p\left(\frac{f_{X^n}^{(k)}(X^n)}{f_{X^n}(X^n)}\right) = E_{P_{X^n}}\left(\frac{f_{X^n}^{(k)}}{f_{X^n}}\right) = \int \left(\frac{f_{X^n}^{(k)}}{f_{X^n}}\right) dP_{X^n}.$$

Define the set $A_n = \{x^n : \ f_{X^n} > 0\}$; then $P_{X^n}(A_n) = 1$. Use the fact that $f_{X^n} = dP_{X^n}/dM_{X^n}$ to write

$$E_P\left(\frac{f_{X^n}^{(k)}(X^n)}{f_{X^n}(X^n)}\right) = \int_{A_n} \left(\frac{f_{X^n}^{(k)}}{f_{X^n}}\right) f_{X^n}\, dM_{X^n} = \int_{A_n} f_{X^n}^{(k)}\, dM_{X^n}.$$

From Theorem 10.1,

$$f_{X^n}^{(k)} = \frac{dP_{X^n}^{(k)}}{dM_{X^n}}$$

and therefore

$$E_p\left(\frac{f_{X^n}^{(k)}(X^n)}{f_{X^n}(X^n)}\right) = \int_{A_n} \frac{dP_{X^n}^{(k)}}{dM_{X^n}} dM_{X^n} = P_{X^n}^{(k)}(A_n) \leq 1.$$

Thus we can apply Lemma 7.13 to the sequence $f_{X^n}^{(k)}(X^n)/f_{X^n}(X^n)$ to conclude that with p-probability 1

$$\lim_{n\to\infty} \frac{1}{n} \ln \frac{f_{X^n}^{(k)}(X^n)}{f_{X^n}(X^n)} \leq 0$$

and hence

$$\lim_{n\to\infty} \frac{1}{n} \ln f_{X^n}^{(k)}(X^n) \leq \liminf_{n\to\infty} \frac{1}{n} f_{X^n}(X^n). \tag{11.1}$$

The left-hand limit is well-defined by the usual ergodic theorem:

$$\lim_{n\to\infty} \frac{1}{n} \ln f_{X^n}^{(k)}(X^n) = \lim_{n\to\infty} \frac{1}{n} \sum_{l=k}^{n-1} \ln f_{X_l|X_{l-k}^k}(X_l|X_{l-k}^k) + \lim_{n\to\infty} \frac{1}{n} \ln f_{X^k}(X^k).$$

Since $0 < f_{X^k} < \infty$ with probability 1 under M_{X^k} and hence also under P_{X^k}, then $0 < f_{X^k}(X^k) < \infty$ under p and therefore $n^{-1} \ln f_{X^k}(X^k) \to 0$ as $n \to \infty$ with probability one. Furthermore, from the pointwise ergodic theorem for stationary and ergodic processes (e.g., Theorem 7.2.1 of [55] or Theorem 8.1 of [58]), since p is stationary ergodic we have with

probability one under p using (10.19) and Lemma 10.1 that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \sum_{l=k}^{n-1} \ln f_{X_l|X_{l-k}^k}(X_l|X_{l-k}^k) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{l=k}^{n-1} \ln f_{X_0|X_{-1},\cdots,X_{-k}}(X_0 \mid X_{-1},\cdots,X_{-k})T^l \\
&= E_p \ln f_{X_0|X_{-1},\cdots,X_{-k}}(X_0|X_{-1},\cdots,X_{-k}) \\
&= H_{p\|m}(X_0|X_{-1},\cdots,X_{-k}) = \overline{H}_{p^{(k)}\|m}(X).
\end{aligned}$$

Thus with (11.1) it follows that

$$\liminf_{n\to\infty} \frac{1}{n} \ln f_{X^n}(X^n) \ge H_{p\|m}(X_0|X_{-1},\cdots,X_{-k}) \tag{11.2}$$

for any positive integer k. Since m is Kth order Markov, Lemma 10.1 and the above imply that

$$\liminf_{n\to\infty} \frac{1}{n} \ln f_{X^n}(X^n) \ge H_{p\|m}(X_0|X^-) = \overline{H}_{p\|m}(X), \tag{11.3}$$

which completes half of the sandwich proof of the theorem.

If $\overline{H}_{p\|m}(X) = \infty$, the proof is completed with (11.3). Hence we can suppose that $\overline{H}_{p\|m}(X) < \infty$. From Lemma 10.1 using the distribution $S_{X_0,X_{-1},X_{-}2,\cdots}$ constructed there, we have that

$$D(P_{X_0,X_{-1},\cdots}\|S_{X_0,X_{-1},\cdots}) = H_{p\|m}(X_0|X^-) = \int dP_{X_0,X^-} \ln f_{X_0|X^-}$$

where

$$f_{X_0|X^-} = \frac{dP_{X_0,X_{-1},\cdots}}{dS_{X_0,X_{-1},\cdots}}.$$

It should be pointed out that we have not (and will not) prove that $f_{X_0|X_{-1},\cdots,X_{-}n} \to f_{X_0|X^-}$; the convergence of conditional probability densities which follows from the martingale convergence theorem and the result about which most generalized Shannon-McMillan-Breiman theorems are built. (See, e.g., Barron [8].) We have proved, however, that the expectations converge (Lemma 10.1), which is what is needed to make the sandwich argument work.

For the second half of the sandwich proof we construct a measure Q which will be dominated by p on semi-infinite sequences using the above conditional densities given the infinite past. Define the semi-infinite sequence

$$X_n^- = \{\cdots, X_{n-1}\}$$

for all nonnegative integers n. Let $\mathcal{B}_k^n = \sigma(X_k^n)$ and $\mathcal{B}_k^- = \sigma(X_k^-) = \sigma(\cdots, X_{k-1})$ be the σ-fields generated by the finite dimensional random

vector X_k^n and the semi-infinite sequence X_k^-, respectively. Let Q be the process distribution having the same restriction to $\sigma(X_k^-)$ as does p and the same restriction to $\sigma(X_0, X_1, \cdots)$ as does p, but which makes X^- and X_k^n conditionally independent given X^k for any n; that is,

$$Q_{X_k^-} = P_{X_k^-},$$

$$Q_{X_k, X_{k+1}, \cdots} = P_{X_k, X_{k+1}, \cdots},$$

and $X^- \to X^k \to X_k^n$ is a Markov chain for all positive integers n so that

$$Q(X_k^n \in F | X_k^-) = Q(X_k^n \in F | X^k).$$

The measure Q is a (nonstationary) k-step Markov approximation to P in the sense of Section 7.2 and

$$Q = P_{X^- \times (X_k, X_{k+1}, \cdots) | X^k}$$

(in contrast to $P = P_{X^- X^k X_k^\infty}$). Observe that $X^- \to X^k \to X_k^n$ is a Markov chain under both Q and m.

By assumption,

$$H_{p\|m}(X_0 | X^-) < \infty$$

and hence from Lemma 10.1

$$H_{p\|m}(X_k^n | X_k^-) = n H_{p\|m}(X_k^n | X_k^-) < \infty$$

and hence from Theorem 7.3 the density $f_{X_k^n | X_k^-}$ is well-defined as

$$f_{X_k^n | X_k^-} = \frac{dS_{X_{n+k}^-}}{P_{X_{n+k}^-}} \tag{11.4}$$

where

$$S_{X_{n+k}^-} = \overline{M_{X_k^n | X^k} P_{X_k^-}}, \tag{11.5}$$

and

$$\begin{aligned} \int dP_{X_{n+k}^-} \ln f_{X_k^n | X_k^-} &= D(P_{X_{n+k}^-} \| S_{X_{n+k}^-}) \\ &= n H_{p\|m}(X_k^n | X_k^-) < \infty. \end{aligned}$$

Thus, in particular,

$$S_{X_{n+k}^-} \gg P_{X_{n+k}^-}.$$

Consider now the sequence of ratios of conditional densities

$$\zeta_n = \frac{f_{X_k^n | X^k}(X^{n+k})}{f_{X_k^n | X_k^-}(X_{n+k}^-)}.$$

We have that

$$\int dp\zeta_n = \int_{G_n} \zeta_n$$

where

$$G_n = \{x : f_{X_k^n|X_k^-}(\mathbf{x}_{n+k}^-) > 0\}$$

since G_n has probability 1 under p (or else (11.6) would be violated). Thus

$$\begin{aligned}
\int dp\zeta_n &= \int dP_{X_{n+k}^-} \left(\frac{f_{X_k^n|X^k}(X^{n+k})}{f_{X_k^n|X_k^-}} 1_{\{f_{X_k^n|X_k^-}>0\}} \right) \\
&= \int dS_{X_{n+k}^-} f_{X_k^n|X_k^-} \left(\frac{f_{X_k^n|X^k}(X^{n+k})}{f_{X_k^n|X_k^-}} 1_{\{f_{X_k^n|X_k^-}>0\}} \right) \\
&= \int dS_{X_{n+k}^-} f_{X_k^n|X^k}(X^{n+k}) 1_{\{f_{X_k^n|X_k^-}>0\}} \\
&\le \int dS_{X_{n+k}^-} f_{X_k^n|X^k}(X^{n+k}).
\end{aligned}$$

Using the definition of the measure S and iterated expectation we have that

$$\begin{aligned}
\int dp\zeta_n &\le \int dM_{X_k^n|X_k^-} dP_{X_k^-} f_{X_k^n|X^k}(X^{n+k}) \\
&= \int dM_{X_k^n|X^k} dP_{X_k^-} f_{X_k^n|X^k}(X^{n+k}).
\end{aligned}$$

Since the integrand is now measurable with respect to $\sigma(X^{n+k})$, this reduces to

$$\int dp\zeta_n \le \int dM_{X_k^n|X^k} dP_{X^k} f_{X_k^n|X^k}.$$

Applying Lemma 7.10 we have

$$\begin{aligned}
\int dp\zeta_n &\le \int dM_{X_k^n|X^k} dP_{X^k} \frac{dP_{X_k^n|X^k}}{dM_{X_k^n|X^k}} \\
&= \int dP_{X^k} dP_{X_k^n|X^k} = 1.
\end{aligned}$$

Thus

$$\int dp\zeta_n \le 1$$

and we can apply Lemma 7.12 to conclude that p-a.e.

$$\limsup_{n\to\infty} \zeta_n = \limsup_{n\to\infty} \frac{1}{n} \ln \frac{f_{X_k^n|X^k}}{f_{X_k^n|X_k^-}} \le 0. \tag{11.6}$$

Using the chain rule for densities,

$$\frac{f_{X_k^n|X^k}}{f_{X_k^n|X_k^-}} = \frac{f_{X^n}}{f_{X^k}} \times \frac{1}{\prod_{l=k}^{n-1} f_{X_l|X_l^-}}.$$

Thus from (11.6)

$$\limsup_{n\to\infty} \left(\frac{1}{n} \ln f_{X^n} - \frac{1}{n} \ln f_{X^k} - \frac{1}{n} \sum_{l=k}^{n-1} \ln f_{X_l|X_l^-} \right) \le 0.$$

Invoking the ergodic theorem for the rightmost terms and the fact that the middle term converges to 0 almost everywhere since $\ln f_{X^k}$ is finite almost everywhere implies that

$$\limsup_{n\to\infty} \frac{1}{n} \ln f_{X^n} \le E_p(\ln f_{X_k|X_k^-}) = E_p(\ln f_{X_0|X^-}) = \overline{H}_{p\|m}(X). \tag{11.7}$$

Combining this with (11.3) completes the sandwich and proves the theorem. □

11.2 Stationary Nonergodic Sources

Next suppose that the source p is stationary with ergodic decomposition $\{p_\lambda;\ \lambda \in \Lambda\}$ and ergodic component function ψ as in Theorem 1.6. We first require some technical details to ensure that the various Radon-Nikodym derivatives are well-defined and that the needed chain rules for densities hold.

Lemma 11.1. *Given a stationary source $\{X_n\}$, let $\{p_\lambda;\ \lambda \in \Lambda\}$ denote the ergodic decomposition and ψ the ergodic component function of Theorem 1.6. Let P_ψ denote the induced distribution of ψ. Let P_{X^n} and $P_{X^n}^\lambda$ denote the induced marginal distributions of p and p_λ. Assume that $\{X_n\}$ has the finite-gap information property of (8.15); that is, there exists a K such that*

$$I_p(X_K; X^-|X^K) < \infty, \tag{11.8}$$

where $X^- = (X_{-1}, X_{-2}, \cdots)$. We also assume that for some n

$$I(X^n; \psi) < \infty. \tag{11.9}$$

This will be the case, for example, if (11.8) holds for $K = 0$. Let m be a K-step Markov process such that $M_{X^n} \gg P_{X^n}$ for all n. (Observe that such a process exists since from (11.8) the Kth order Markov approximation $p^{(K)}$ suffices.) Define $M_{X^n,\psi} = M_{X^n} \times P_\psi$. Then

$$M_{X^n,\psi} \gg P_{X^n} \times P_\psi \gg P_{X^n,\psi}, \tag{11.10}$$

and with probability 1 under p

$$M_{X^n} \gg P_{X^n} \gg P^\psi_{X^n}.$$

Lastly,

$$\frac{dP^\psi_{X^n}}{dM_{X^n}} = f_{X^n|\psi} = \frac{dP_{X^n,\psi}}{d(M_{X^n} \times P_\psi)}. \tag{11.11}$$

and therefore

$$\frac{dP^\psi_{X^n}}{dP_{X^n}} = \frac{dP^\psi_{X^n}/dM_{X^n}}{dP_{X^n}/dM_{X^n}} = \frac{f_{X^n|\psi}}{f_{X^n}}. \tag{11.12}$$

Proof: From Theorem 8.5 the given assumptions ensure that

$$\lim_{n\to\infty} \frac{1}{n} E_p i(X^n;\psi) = \lim_{n\to\infty} \frac{1}{n} I(X^n;\psi) = 0 \tag{11.13}$$

and hence $P_{X^n} \times P_\psi \gg P_{X^n,\psi}$ (since otherwise $I(X^n;\psi)$ would be infinite for some n and hence infinite for all larger n since it is increasing with n). This proves the right-most absolute continuity relation of (11.10). This in turn implies that $M_{X^n} \times P_\psi \gg P_{X^n,\psi}$. The lemma then follows from Theorem 7.2 with $X = X^n$, $Y = \psi$ and the chain rule for Radon-Nikodym derivatives. □

We know that the source will produce with probability one an ergodic component p_λ and hence Theorem 11.1 will hold for this ergodic component. In other words, we have for all λ that

$$\lim_{n\to\infty} \frac{1}{n} \ln f_{X^n|\psi}(X^n|\lambda) = \overline{H}_{p_\lambda}(X);\ p_\lambda - \text{a.e.}$$

This implies that

$$\lim_{n\to\infty} \frac{1}{n} \ln f_{X^n|\psi}(X^n|\psi) = \overline{H}_{p_\psi}(X);\ p - \text{a.e.} \tag{11.14}$$

Making this step precise generalizes Lemma 4.3.

Lemma 11.2. *Suppose that $\{X_n\}$ is a stationary not necessarily ergodic source with ergodic component function ψ. Then (11.14) holds.*

Proof: The proof parallels that for Lemma 4.3. Observe that if we have two random variables U, V ($U = X_0, X_1, \cdots$ and $Y = \psi$ above) and a sequence of functions $g_n(U,V)$ ($n^{-1} f_{X^n|\psi}(X^n|\psi)$) and a function $g(V)$ ($\overline{H}_{p_\psi}(X)$) with the property

$$\lim_{n\to\infty} g_n(U,v) = g(v), P_{U|V=v} - \text{a.e.},$$

then also

$$\lim_{n\to\infty} g_n(U,V) = g(V);\ P_{UV} - \text{a.e.}$$

since defining the (measurable) set $G = \{u,v : \lim_{n\to\infty} g_n(u,v) = g(v)\}$ and its section $G_v = \{u : (u,v) \in G\}$, then from (1.28)

$$P_{UV}(G) = \int P_{U|V}(G_v|v)dP_V(v) = 1$$

if $P_{U|V}(G_v|v) = 1$ with probability 1. □

It is not, however, the relative entropy density using the distribution of the ergodic component that we wish to show converges. It is the original sample density f_{X^n}. The following lemma shows that the two sample entropies converge to the same thing. The lemma generalizes Lemma 4.3 and is proved by a sandwich argument analogous to Theorem 11.1. The result can be viewed as an almost everywhere version of (11.13).

Theorem 11.2. *Given a stationary source $\{X_n\}$, let $\{p_\lambda;\ \lambda \in \Lambda\}$ denote the ergodic decomposition and ψ the ergodic component function of Theorem 1.6. Assume that the finite-gap information property (11.8) is satisfied and that (11.9) holds for some n. Then*

$$\lim_{n\to\infty} \frac{1}{n} i(X^n;\psi) = \lim_{n\to\infty} \frac{1}{n} \ln \frac{f_{X^n|\psi}}{f_{X^n}} = 0;\ p - \text{a.e.}$$

Proof: From Theorem 7.4 we have immediately that

$$\liminf_{n\to\infty} i_n(X^n;\psi) \ge 0, \tag{11.15}$$

which provides half of the sandwich proof.

To develop the other half of the sandwich, for each $k \ge K$ let $p^{(k)}$ denote the k-step Markov approximation of p. Exactly as in the proof of Theorem 11.1, it follows that (11.1) holds. Now, however, the Markov approximation relative entropy density converges instead as

$$\lim_{n\to\infty} \frac{1}{n} \ln f^{(k)}_{X^n}(X^n) = \lim_{n\to\infty} \frac{1}{n} \sum_{l=k}^{\infty} f_{X_k|X^k}(X_k|X^k)T^k = E_{p_\psi} f_{X_k|X^k}(X_k|X^k).$$

Combining this with (11.14 we have that

$$\limsup_{n\to\infty} \frac{1}{n} \ln \frac{f_{X^n|\psi}(X^n|\psi)}{f_{X^n}(X^n)} \le \overline{H}_{p_\psi\|m}(X) - E_{p_\psi} f_{X_k|X^k}(X_k|X^k).$$

From Lemma 10.1, the right hand side is just $I_{p_\psi}(X_k;X^-|X^k)$ which from Corollary 10.4 is just $\overline{H}_{p\|p^{(k)}}(X)$. Since the bound holds for all k, we have that

$$\limsup_{n\to\infty} \frac{1}{n} \ln \frac{f_{X^n|\psi}(X^n|\psi)}{f_{X^n}(X^n)} \le \inf_k \overline{H}_{p_\psi \| p^{(k)}}(X) \equiv \zeta.$$

Using the ergodic decompostion of relative entropy rate (Corollary 10.7) that and the fact that Markov approximations are asymptotically accurate (Corollary 10.5) we have further that

$$\begin{aligned} \int dP_\psi \zeta &= \int dP_\psi \inf_k \overline{H}_{p_\psi \| p^{(k)}}(X) \\ &\le \inf_k \int dP_\psi \overline{H}_{p_\psi \| p^{(k)}}(X) \\ &= \inf_k \overline{H}_{p \| p^{(k)}}(X) = 0 \end{aligned}$$

and hence $\zeta = 0$ with P_ψ probability 1. Thus

$$\limsup_{n\to\infty} \frac{1}{n} \ln \frac{f_{X^n|\psi}(X^n|\psi)}{f_{X^n}(X^n)} \le 0, \tag{11.16}$$

which with (11.15) completes the sandwich proof. □

Simply restating the theorem yields and using (11.14) the ergodic theorem for relative entropy densities in the general stationary case.

Corollary 8.3.1: Given the assumptions of Theorem 11.2,

$$\lim_{n\to\infty} \frac{1}{n} \ln f_{X^n}(X^n) = \overline{H}_{p_\psi \| m}(X), p - \text{a.e.}$$

The corollary states that the sample relative entropy density of a process satisfying (11.8) converges to the conditional relative entropy rate with respect to the underlying ergodic component. This is a slight extension and elaboration of Barron's result [8] which made the stronger assumption that $H_{p\|m}(X_0|X^-) = \overline{H}_{p\|m}(X) < \infty$. From Corollary 10.5 this condition is sufficient but not necessary for the finite-gap information property of (11.8). In particular, the finite gap information property implies that

$$\overline{H}_{p\|p^{(k)}}(X) = I_p(X_k; X^-|X^k) < \infty,$$

but it need not be true that $\overline{H}_{p\|m}(X) < \infty$. In addition, Barron [8] and Algoet and Cover [7] do not characterize the limiting density as the entropy rate of the ergodic component, instead they effectively show that the limit is $E_{p_\psi}(\ln f_{X_0|X^-}(X_0|X^-))$. This, however, is equivalent since it follows from the ergodic decomposition (see specifically Lemma 8.6.2 of [55] or Lemma 10.4 of [58]) that $f_{X_0|X^-} = f_{X_0|X^-,\psi}$ with probability one since the ergodic component ψ can be determined from the infinite past X^-.

11.3 AMS Sources

The following lemma is a generalization of Lemma 4.5. The result is due to Barron [8], who proved it using martingale inequalities and convergence results.

Lemma 11.3. *Let $\{X_n\}$ be an AMS source with the property that for every integer k there exists an integer $l = l(k)$ such that*

$$I_p(X^k; (X_{k+l}, X_{k+l+1}, \cdots)|X_k^l) < \infty. \tag{11.17}$$

Then

$$\lim_{n\to\infty} \frac{1}{n} i(X^k; (X_k + l, \cdots, X_{n-1})|X_k^l) = 0; \ p - \text{a.e.}$$

Proof: By assumption

$$I_p(X^k; (X_{k+l}, X_{k+l+1}, \cdots)|X_k^l) = E_p \ln \frac{f_{X^k|X_k,X_{k+1},\cdots}(X^k|X_k, X_{k+1}, \cdots)}{f_{X^k|X_k^l}(X^k|X_k^l)} < \infty.$$

This implies that

$$P_{X^k\times(X_k+l,\cdots)|X_k^l} \gg P_{X_0,X_1,\cdots}$$

with

$$\frac{dP_{X_0,X_1,\cdots}}{dP_{X^k\times(X_k+l,\cdots)|X_k^l}} = \frac{f_{X^k|X_k,X_k+1,\cdots}(X^k|X_k, X_k + 1, \cdots)}{f_{X^k|X_k^l}(X^k|X_k^l).}.$$

Restricting the measures to X^n for $n > k + l$ yields

$$\begin{aligned}\frac{dP_{X^n}}{dP_{X^k\times(X_k+l,\cdots,X^n)|X_k^l}} &= \frac{f_{X^k|X_k,X_k+1,\cdots,X_n}(X^k|X_k, X_k + 1, \cdots)}{f_{X^k|X_k^l}(X^k|X_k^l)} \\ &= i(X^k; (X_k + l, \cdots, X_n)|X_k^l).\end{aligned}$$

With this setup the lemma follows immediately from Theorem 7.4. □

The following lemma generalizes Lemma 4.6 and will yield the general theorem. The lemma was first proved by Barron [8] using martingale inequalities.

Theorem 11.3. *Suppose that p and m are distributions of a standard alphabet process $\{X_n\}$ such that p is AMS and m is k-step Markov. Let $\overline{p}$ be a stationary measure that asymptotically dominates p (e.g., the stationary mean). Suppose that P_{X^n}, $\overline{P}_{X^n}$, and M_{X^n} are the distributions induced by p, $\overline{p}$, and m and that M_{X^n} dominates both P_{X^n} and $\overline{P}_{X^n}$ for all n and that f_{X^n} and $\overline{f}_{X^n}$ are the corresponding densities. If there is an invariant function h such that*

$$\lim_{n\to\infty} \frac{1}{n} \ln \overline{f}_{X^n}(X^n) = h;\ \overline{p} - \text{a.e.}$$

then also

$$\lim_{n\to\infty} \frac{1}{n} \ln f_{X^n}(X^n) = h;\ p - \text{a.e.}$$

Proof: For any k and $n \geq k$ we can write using the chain rule for densities

$$\frac{1}{n} \ln f_{X^n} - \frac{1}{n} \ln f_{X_k^{n-k}} = \frac{1}{n} \ln f_{X^k|X_k^{n-k}}.$$

Since for $k \leq l < n$

$$\frac{1}{n} \ln f_{X^k|X_k^{n-k}} = \frac{1}{n} \ln f_{X^k|X_k^l} + \frac{1}{n} i(X^k; (X_{k+l}, \cdots, X_{n-1})|X_k^l),$$

Lemma 11.3 and the fact that densities are finite with probability one implies that

$$\lim_{n\to\infty} \frac{1}{n} \ln f_{X^k|X_k^{n-k}} = 0;\ p - \text{a.e.}$$

This implies that there is a subsequence $k(n) \to \infty$ such that

$$\frac{1}{n} \ln f_{X^n}(X^n) - \frac{1}{n} \ln f_{X_{k(n)}^{n-k(n)})}(X_{k(n)}^{n-k(n)});\ \to 0, p - \text{a.e.}$$

To prove this, for each k chose $N(k)$ large enough so that

$$p(|\frac{1}{N(k)} \ln f_{X^k|X_k^{N(k)-k}}(X^k|X_k^{N(k)-k})| > 2^{-k}) \leq 2^{-k}$$

and then let $k(n) = k$ for $N(k) \leq n < N(k+1)$. Then from the Borel-Cantelli lemma we have for any ϵ that

$$p(|\frac{1}{N(k)} \ln f_{X^k|X_k^{N(k)-k}}(X^k|X_k^{N(k)-k})| > \epsilon \text{ i.o.}) = 0$$

and hence

$$\lim_{n\to\infty} \frac{1}{n} \ln f_{X^n}(X^n) = \lim_{n\to\infty} \frac{1}{n} \ln f_{X_{k(n)}^{n-k(n)}}(X_{k(n)}^{n-k(n)});\ p - \text{a.e.}$$

In a similar manner we can also choose the sequence so that

$$\lim_{n\to\infty} \frac{1}{n} \ln \overline{f}_{X^n}(X^n) = \lim_{n\to\infty} \frac{1}{n} \ln \overline{f}_{X_{k(n)}^{n-k(n)}}(X_{k(n)}^{n-k(n)});\ \overline{p} - \text{a.e.}$$

From Markov's inequality

$$\overline{p}\left(\frac{1}{n}\ln f_{X_k^{n-k}}(X_k^{n-k}) \geq \frac{1}{n}\ln \overline{f}_{X_k^{n-k}}(X_k^{n-k}) + \epsilon\right)$$
$$= \overline{p}\left(\frac{f_{X_k^{n-k}}(X_k^{n-k})}{\overline{f}_{X_k^{n-k}}(X_k^{n-k})} \geq e^{n\epsilon}\right)$$
$$\leq e^{-n\epsilon}\int d\overline{p}\frac{f_{X_k^{n-k}}(X_k^{n-k})}{\overline{f}_{X_k^{n-k}}(X_k^{n-k})} = e^{-n\epsilon}\int dm f_{X_k^{n-k}}(X_k^{n-k}) = e^{-n\epsilon}.$$

Hence again invoking the Borel-Cantelli lemma we have that

$$\overline{p}\left(\frac{1}{n}\ln f_{X_k^{n-k}}(X_k^{n-k}) \geq \frac{1}{n}\ln \overline{f}_{X_k^{n-k}}(X_k^{n-k}) + \epsilon \text{ i.o.}\right) = 0$$

and therefore

$$\limsup_{n\to\infty}\frac{1}{n}\ln f_{X_k^{n-k}}(X_k^{n-k}) \leq h, \overline{p}-\text{a.e.} \tag{11.18}$$

The above event is in the tail σ-field $\bigcap_n \sigma(X_n, X_{n+1}, \cdots)$ since h is invariant and $\overline{p}$ dominates p on the tail σ-field. Thus

$$\limsup_{n\to\infty}\frac{1}{n}\ln f_{X_{k(n)}^{n-k(n)}}(X_{k(n)}^{n-k(n)}) \leq h;\ p-\text{a.e.}$$

and hence

$$\limsup_{n\to\infty}\frac{1}{n}\ln f_{X^n}(X^n) \leq h;\ p-\text{a.e.}$$

which proves half of the lemma.

Since $\overline{p}$ asymptotically dominates p, given $\epsilon > 0$ there is a k such that

$$p\left(\lim_{n\to\infty} n^{-1}\overline{f}(X_k^{n-k}) = h\right) \geq 1-\epsilon.$$

Again applying Markov's inequality and the Borel-Cantelli lemma as previously we have that

$$\liminf_{n\to\infty}\frac{1}{n}\ln\frac{f_{X_{k(n)}^{n-k(n)}}(X_{k(n)}^{n-k(n)})}{\overline{f}_{X_{k(n)}^{n-k(n)}}(X_{k(n)}^{n-k(n)})} \geq 0;\ p-\text{a.e.}$$

which implies that

$$p\left(\liminf_{n\to\infty}\frac{1}{n}f_{X_{k(n)}^{n-k(n)}}(X_k^{n-k}) \geq h\right) \geq \epsilon$$

and hence also that

$$p\left(\liminf_{n\to\infty}\frac{1}{n}f_{X^n}(X^n) \geq h\right) \geq \epsilon.$$

Since ϵ can be made arbitrarily small, this proves that p-a.e. $\liminf n^{-1} h_n \geq h$, which completes the proof of the lemma. □

We can now extend the ergodic theorem for relative entropy densities to the general AMS case.

Corollary 8.4.1: Given the assumptions of Theorem 11.3,

$$\lim_{n\to\infty} \frac{1}{n} \ln f_{X^n}(X^n) = \overline{H}_{\overline{p}_\psi}(X),$$

where $\overline{p}_\psi$ is the ergodic component of the stationary mean $\overline{p}$ of p.

Proof: The proof follows immediately from Theorem 11.3 and Lemma 11.1, the ergodic theorem for the relative entropy density for the stationary mean. □

11.4 Ergodic Theorems for Information Densities.

As an application of the general theorem we prove an ergodic theorem for mutual information densities for stationary and ergodic sources. The result can be extended to AMS sources in the same manner that the results of Section 11.2 were extended to those of Section 11.3. As the stationary and ergodic result suffices for the coding theorems and the AMS conditions are messy, only the stationary case is considered here. The result is due to Barron [8].

Theorem 11.4. *Let $\{X_n, Y_n\}$ be a stationary ergodic pair random process with standard alphabet. Let $P_{X^nY^n}$, P_{X^n}, and P_{Y^n} denote the induced distributions and assume that for all n $P_{X^n} \times P_{Y^n} \gg P_{X^nY^n}$ and hence the information densities*

$$i_n(X^n; Y^n) = \frac{dP_{X^nY^n}}{d(P_{X^n} \times P_{Y^n})}$$

are well-defined. Assume in addition that both the $\{X_n\}$ and $\{Y_n\}$ processes have the finite-gap information property of (11.8) and hence by the comment following Corollary 10.1 there is a K such that both processes satisfy the K-gap property

$$I(X_K; X^- | X^K) < \infty, \; I(Y_K; Y^- | Y^K) < \infty.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} i_n(X^n; Y^n) = \bar{I}(X; Y); \; p - \text{a.e.}.$$

Proof: Let $Z_n = (X_n, Y_n)$. Let $M_{X^n} = P^{(K)}_{X^n}$ and $M_{Y^n} = P^{(K)}_{Y^n}$ denote the Kth order Markov approximations of $\{X_n\}$ and $\{Y_n\}$, respectively. The finite-gap approximation implies as in Section 11.2 that the densities

$$f_{X^n} = \frac{dP_{X^n}}{dM_{X^n}} \text{ and } f_{Y^n} = \frac{dP_{Y^n}}{dM_{Y^n}}$$

are well-defined. From Theorem 11.1

$$\lim_{n\to\infty} \frac{1}{n} \ln f_{X^n}(X^n) = H_{p_X \| p_X^{(K)}}(X_0|X^-) = I(X_k; X^-|X^k) < \infty,$$
$$\lim_{n\to\infty} \frac{1}{n} \ln f_{Y^n}(Y^n) = I(Y_k; Y^-|Y^k) < \infty.$$

Define the measures M_{Z^n} by $M_{X^n} \times M_{Y^n}$. Then this is a K-step Markov source and since

$$M_{X^n} \times M_{Y^n} \gg P_{X^n} \times P_{Y^n} \gg P_{X^n,Y^n} = P_{Z^n},$$

the density

$$f_{Z^n} = \frac{dP_{Z^n}}{dM_{Z^n}}$$

is well-defined and from Theorem 11.1 has a limit

$$\lim_{n\to\infty} \frac{1}{n} \ln f_{Z^n}(Z^n) = H_{p\|m}(Z_0|Z^-).$$

If the density $i_n(X^n, Y^n)$ is infinite for any n, then it is infinite for all larger n and convergence is trivially to the infinite information rate. If it is finite, the chain rule for densities yields

$$\begin{aligned}
\frac{1}{n} i_n(X^n; Y^n) &= \frac{1}{n} \ln f_{Z^n}(Z^n) - \frac{1}{n} \ln f_{X^n}(X^n) - \frac{1}{n} \ln f_{Y^n}(Y^n) \\
&\underset{n\to\infty}{\to} H_{p\|p^{(k)}}(Z_0|Z^-) - H_{p\|p^{(k)}}(X_0|X^-) - H_{p\|p^{(k)}}(Y_0|Y^-) \\
&= \overline{H}_{p\|p^{(k)}}(X, Y) - \overline{H}_{p\|p^{(k)}}(X) - \overline{H}_{p\|p^{(k)}}(Y).
\end{aligned}$$

The limit is not indeterminate (of the form $\infty - \infty$) because the two subtracted terms are finite. Since convergence is to a constant, the constant must also be the limit of the expected values of $n^{-1} i_n(X^n, Y^n)$, that is, $\overline{I}(X; Y)$. □

Chapter 12
Source Coding Theorems

Abstract The source coding theorems subject to a fidelity criterion are develped for AMS sources and additive and subadditive distortion measures. The results are first developed for the classic case of block coding and then to sliding-block codes. The operational distortion-rate function for both classes of codes is shown to equal the Shannon distortion-rate function.

12.1 Source Coding and Channel Coding

In this chapter and in Chapter 14 we develop the basic coding theorems of information theory. As is traditional, we consider two important special cases first and then later form the overall result by combining these special cases. In the first case in this chapter we assume that the channel is noiseless, but it is constrained in the sense that it can only pass R bits per input symbol to the receiver. Since this is usually insufficient for the receiver to perfectly recover the source sequence, we attempt to code the source so that the receiver can recover it with as little distortion as possible. This leads to the theory of *source coding* or *source coding subject to a fidelity criterion* or *data compression*, where the latter name reflects the fact that sources with infinite or very large entropy are "compressed" to fit across the given communication link. In Chapter 14 we ignore the source and focus on a discrete alphabet channel and construct codes that can communicate any of a finite number of messages with small probability of error and we quantify how large the message set can be. This operation is called *channel coding* or *error control coding*. We then develop *joint source and channel codes* which combine source coding and channel coding so as to code a given source for communication over a given channel so as to minimize average distortion. The *ad hoc* division into two forms of coding is convenient and will permit performance near

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that of the operational distortion-rate function function for the single-user or point-to-point communication systems and codes considered in this book.

The Shannon coding theorems quantify the optimal performance that can be achieved when communicating a given source through a given channel, but they do not say how to actually achieve such optimal performance. The Shannon theorems are at heart existence theorems and not constructive. There is a huge literature on constructing channel codes for reliable communication and source codes for analog-to-digital conversion and data compression. Coding theory in the sense of a rigorous approach to designing good codes is not treated or even surveyed here, but there are a collection of results which use the information theoretic techniques developed in this book to provide necessary conditions that optimal or asymptotically optimal source codes must satisfy. Such conditions highlight implications of the underlying theory for the behavior of good codes and provide insight into the structure of good codes and thereby suggest design techniques that can improve existing codes. These results are developed in Chapter 13.

12.2 Block Source Codes for AMS Sources

We first consider a particular class of codes: block codes. For the time being we also concentrate on additive distortion measures. Extensions to subadditive distortion measures will be considered later. Let $\{X_n\}$ be a source with a standard alphabet A. Recall that an (N, K) block code of a source $\{X_n\}$ maps successive nonoverlapping input vectors $\{X_{nN}^N\}$ into successive channel vectors $U_{nK}^K = \alpha(X_{nN}^N)$, where $\alpha : A^N \to B^K$ is called the *source encoder*. We assume that the channel is noiseless, but that it is constrained in the sense that N source time units corresponds to the same amount of physical time as K channel time units and that

$$\frac{K \log ||B||}{N} \le R,$$

where the inequality can be made arbitrarily close to equality by taking N and K large enough subject to the physical stationarity constraint. R is called the *source coding rate* or *resolution* in bits or nats per input symbol. We may wish to change the values of N and K, but the rate is fixed.

A reproduction or approximation of the original source is obtained by a *source decoder*, which we also assume to be a block code. The decoder is a mapping $\alpha : B^K \to \hat{A}^N$ which forms the reproduction process $\{\hat{X}_n\}$ via $\hat{X}_{nN}^N = \alpha(U_{nK}^K)$; $n = 1, 2, \ldots$. In general we could have a reproduction dimension different from that of the input vectors provided they corre-

sponded to the same amount of physical time and a suitable distortion measure was defined. We will make the simplifying assumption that they are the same, however.

Because N source symbols are mapped into N reproduction symbols, we will often refer to N alone as the block length of the source code. Observe that the resulting sequence coder is N-stationary. Our immediate goal is now the following: Let $\mathcal{E}$ and $\mathcal{D}$ denote the collection of all block codes with rate no greater than R and let ν be the given channel. What is the optimal achievable performance $\Delta(\mu, \mathcal{E}, \nu, \mathcal{D})$ for this system? Our first step toward evaluating the operational DRF is to find a simpler and equivalent expression for the current special case.

Given a source code consisting of encoder α and decoder β, define the *codebook* to be

$$C = \{ \text{all } \beta(u^K); u^K \in B^K \},$$

that is, the collection of all possible reproduction vectors available to the receiver. For convenience we can index these words as

$$C = \{y_i; i = 1, 2, \ldots, M\},$$

where $N^{-1} \log M \le R$ by construction. Observe that if we are given only a decoder β or, equivalently, a codebook, and if our goal is to minimize the average distortion for the current block, then no encoder can do better than the encoder α^* which maps an input word x^N into the minimum distortion available reproduction word, that is, define $\alpha^*(x^N)$ to be the u^K minimizing $\rho_N(x^N, \beta(u^K))$, an assignment we denote by

$$\alpha^*(x^N) = \operatorname*{argmin}_{u^K} \rho_N(x^N, \beta(u^K)).$$

The fact that no encoder can yield smaller average distortion than a minimum distortion encoder is an example of an *optimality property* of block codes. Such properties are the subject of Chapter 13. Observe that by construction we therefore have that

$$\rho_N(x^N, \beta(\alpha^*(x^N))) = \min_{y \in C} \rho_N(x^N, y)$$

and the overall mapping of x^N into a reproduction is a minimum distortion or nearest neighbor mapping. Define

$$\rho_N(x^N, C) = \min_{y \in C} \rho_N(x^N, y).$$

To prove that this is the best encoder, observe that if the source μ is AMS and p is the joint distribution of the source and reproduction, then p is also AMS. This follows since the channel induced by the block code is N-stationary and hence also AMS with respect to T^N. This means that

p is AMS with respect to T^N which in turn implies that it is AMS with respect to T (Theorem 7.3.1 of [55] or Theorem 8.2 of [58]). Letting $\overline{p}$ denote the stationary mean of p and $\overline{p}_N$ denote the N-stationary mean, we then have from (5.12) that for any block codes with codebook C

$$\Delta = \frac{1}{N} E_{\overline{p}_N} \rho_N(X^N, Y^N) \geq \frac{1}{N} E_{\overline{p}_N} \rho_N(X^N, C),$$

with equality if the minimum distortion encoder is used. For this reason we can confine interest for the moment to block codes specified by a codebook: the encoder produces the index of the minimum distortion codeword for the observed vector and the decoder is a table lookup producing the codeword being indexed. We will be interested later in looking at possibly nonoptimal encoders in order to decouple the encoder from the decoder and characterize the separate effects of encoder and decoder on performance.

A block code of this type is also called a *vector quantizer* or *block quantizer*. Denote the performance of the block code with codebook C on the source μ by

$$\rho(C, \mu) = \Delta = E_p \rho_\infty.$$

Lemma 12.1. *Given an AMS source μ and a block length N code book C, let $\overline{\mu}_N$ denote the N-stationary mean of μ (which exists from Corollary 7.3.1 of [55] or Corollary 8.5 of [58]), let p denote the induced input/output distribution, and let $\overline{p}$ and $\overline{p}_N$ denote its stationary mean and N-stationary mean, respectively. Then*

$$\rho(C, \mu) = E_{\overline{p}} \rho_1(X_0, Y_0) = \frac{1}{N} E_{\overline{p}_N} \rho_N(X^N, Y^N)$$
$$= \frac{1}{N} E_{\overline{\mu}_N} \rho_N(X^N, C) = \rho(C, \overline{\mu}_N).$$

Proof: The first two equalities follow from (5.12), the next from the use of the minimum distortion encoder, the last from the definition of the performance of a block code. □

It need not be true in general that $\rho(C, \mu)$ equal $\rho(C, \overline{\mu})$. For example, if μ produces a single periodic waveform with period N and C consists of a single period, then $\rho(C, \mu) = 0$ and $\rho(C, \overline{\mu}) > 0$. It is the N-stationary mean and not the stationary mean that is most useful for studying an N-stationary code.

We now define the operational distortion-rate function (DRF) for block codes to be

$$\delta(R, \mu) = \Delta(\mu, \nu, \mathcal{E}, \mathcal{D}) = \inf_N \delta_N(R, \mu), \tag{12.1}$$

$$\delta_N(R, \mu) = \inf_{C: \frac{1}{N} \log ||C|| \leq R} \rho(C, \mu), \tag{12.2}$$

where ν is the noiseless channel described earlier and $\mathcal{E}$ and $\mathcal{D}$ are classes of block codes for the channel. $\delta(R,\mu)$ is called the *operational block coding distortion-rate function (DRF).*

Corollary 12.1. *Given an AMS source μ, then for any positive integer N*

$$\delta_N(R,\mu T^{-i}) = \delta_N(R,\overline{\mu}_N T^{-i}); i = 0,1,\ldots,N-1.$$

Proof: For $i = 0$ the result is immediate from the lemma. For $i \neq 0$ it follows from the lemma and the fact that the N-stationary mean of μT^{-i} is $\overline{\mu}_N T^{-i}$ (as is easily verified from the definitions). □

Reference Letters

Many of the source coding results will require a technical condition that is a generalization of the reference letter condition of Theorem 9.1 for stationary sources. An AMS source μ is said to have a *reference letter* $a^* \in \hat{A}$ with respect to a distortion measure $\rho = \rho_1$ on $A \times \hat{A}$ if

$$\sup_n E_{\mu T^{-n}} \rho(X_0, a^*) = \sup_n E_\mu \rho(X_n, a^*) = \rho^* < \infty, \tag{12.3}$$

that is, there exists a letter for which $E_\mu \rho(X^n, a^*)$ is uniformly bounded above. If we define for any k the vector $a^{*k} = (a^*, a^*, \cdots, a^*)$ consisting of k a^*'s, then (12.3) implies that

$$\sup_n E_{\mu T^{-n}} \frac{1}{k} \rho_k(X^k, a^{*k}) \le \rho^* < \infty. \tag{12.4}$$

We assume for convenience that any block code of length N contains the reference vector a^{*N}. This ensures that $\rho_N(x^N, C) \le \rho_N(x^N, a^{*N})$ and hence that $\rho_N(x^N, C)$ is bounded above by a μ-integrable function and hence is itself μ-integrable. This implies that

$$\delta(R,\mu) \le \delta_N(R,\mu) \le \rho^*. \tag{12.5}$$

The reference letter also works for the stationary mean source $\overline{\mu}$ since

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \rho(x_i, a^*) = \rho_\infty(x, \mathbf{a}^*),$$

$\overline{\mu}$-a.e. and μ-a.e., where $\mathbf{a}^*$ denotes an infinite sequence of a^*. Since ρ_∞ is invariant we have from Lemma 6.3.1 of [55] or Lemma 7.5 of [58] and Fatou's lemma (Lemma 4.4.5 of [55] or Lemma 8.5 of [58]) that

$$E_{\overline{\mu}}\rho(X_0, a^*) = E_\mu\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\rho(X_i, a^*)\right)$$
$$\le \liminf_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}E_\mu\rho(X_i, a^*) \le \rho^*.$$

Performance and Distortion-Rate Functions

We next develop several basic properties of the performance and the operational DRFs for block coding AMS sources with additive fidelity criteria.

Lemma 12.2. *Given two sources μ_1 and μ_2 and $\lambda \in (0,1)$, then for any block code C*

$$\rho(C, \lambda\mu_1 + (1-\lambda)\mu_2) = \lambda\rho(C, \mu_1) + (1-\lambda)\rho(C, \mu_2)$$

and for any N

$$\delta_N(R, \lambda\mu_1 + (1-\lambda)\mu_2) \ge \lambda\delta_N(R, \mu_1) + (1-\lambda)\delta_N(R, \mu_2)$$

and

$$\delta(R, \lambda\mu_1 + (1-\lambda)\mu_2) \ge \lambda\delta(R, \mu_1) + (1-\lambda)\delta(R, \mu_2).$$

Thus performance is linear in the source and the operational DRFs are convex $\bigcap$ in R. Lastly,

$$\delta_N(R + \frac{1}{N}, \lambda\mu_1 + (1-\lambda)\mu_2) \le \lambda\delta_N(R, \mu_1) + (1-\lambda)\delta_N(R, \mu_2).$$

Proof: The equality follows from the linearity of expectation since $\rho(C, \mu) = E_\mu\rho(X^N, C)$. The first inequality follows from the equality and the fact that the infimum of a sum is bounded below by the sum of the infima. The next inequality follows similarly. To get the final inequality, let C_i approximately yield $\delta_N(R, \mu_i)$; that is,

$$\rho(C_i, \mu_i) \le \delta_N(R, \mu_i) + \epsilon.$$

Form the union code $C = C_1 \bigcup C_2$ containing all of the words in both of the codes. Then the rate of the code is

$$\begin{aligned}\frac{1}{N}\log||C|| &= \frac{1}{N}\log(||C_1|| + ||C_2||)\\ &\le \frac{1}{N}\log(2^{NR} + 2^{NR})\\ &= R + \frac{1}{N}.\end{aligned}$$

This code yields performance

$$\begin{aligned}&\rho(C, \lambda\mu_1 + (1-\lambda)\mu_2)\\ &= \lambda\rho(C, \mu_1) + (1-\lambda)\rho(C, \mu_2)\\ &\le \lambda\rho(C_1, \mu_1) + (1-\lambda)\rho(C_2, \mu_2)\\ &\le \lambda\delta_N(R, \mu_1) + \lambda\epsilon + (1-\lambda)\delta_N(R, \mu_2) + (1-\lambda)\epsilon.\end{aligned}$$

Since the leftmost term in the above equation can be no smaller than $\delta_N(R + 1/N, \lambda\mu_1 + (1-\lambda)\mu_2)$, the lemma is proved. □

The first and last inequalities in the lemma suggest that δ_N is very nearly an affine function of the source and hence perhaps δ is as well. We will later pursue this possibility, but we are not yet equipped to do so.

Before developing the connection between the distortion-rate functions of AMS sources and those of their stationary mean, we pause to develop some additional properties for operational DRFs in the special case of stationary sources. These results follow Kieffer [91].

Lemma 12.3. *Suppose that μ is a stationary source. Then*

$$\delta(R, \mu) = \lim_{N\to\infty} \delta_N(R, \mu).$$

Thus the infimum over block lengths is given by the limit so that longer codes can do better.

Proof: Fix an N and an $n < N$ and choose codes $C_n \subset \hat{A}^n$ and $C_{N-n} \subset \hat{A}^{N-n}$ for which

$$\begin{aligned}\rho(C_n, \mu) &\le \delta_n(R, \mu) + \frac{\epsilon}{2}\\ \rho(C_{N-n}, \mu) &\le \delta_{N-n}(R, \mu) + \frac{\epsilon}{2}.\end{aligned}$$

Form the block length N code $C = C_n \times C_{N-n}$. This code has rate no greater than R and has distortion

$$\begin{aligned} N\rho(C,\mu) &= E \min_{\mathcal{Y}\in C} \rho_N(X^N,\mathcal{Y}) \\ &= E_{\mathcal{Y}^n\in C_n}\rho_n(X^n,\mathcal{Y}^n) + E_{v^{N-n}\in C_{N-n}}\rho_{N-n}(X_n^{N-n},v^{N-n}) \\ &= E_{\mathcal{Y}^n\in C_n}\rho_n(X^n,\mathcal{Y}^n) + E_{v^{N-n}\in C_{N-n}}\rho_{N-n}(X^{N-n},v^{N-n}) \\ &= n\rho(C_n,\mu) + (N-n)\rho(C_{N-n},\mu) \\ &\le n\delta_n(R,\mu) + (N-n)\delta_{N-n}(R,\mu) + \epsilon, \end{aligned} \tag{12.6}$$

where we have made essential use of the stationarity of the source. Since ϵ is arbitrary and since the leftmost term in the above equation can be no smaller than $N\delta_N(R,\mu)$, we have shown that

$$N\delta_N(R,\mu) \le n\delta_n(R,\mu) + (N-n)\delta_{N-n}(R,\mu)$$

and hence that the sequence $N\delta_N$ is subadditive. The result then follows immediately from Lemma 7.5.1 of [55] or Lemma 8.5.3 of [58]. □

Corollary 12.2. *If μ is a stationary source, then $\delta(R,\mu)$ is a convex $\bigcup$ function of R and hence is continuous for $R > 0$.*

Proof: Pick $R_1 > R_2$ and $\lambda \in (0,1)$. Define $R = \lambda R_1 + (1-\lambda)R_2$. For large n define $n_1 = \lfloor \lambda n \rfloor$ be the largest integer less than λn and let $n_2 = n - n_1$. Pick codebooks $C_i \subset \hat{A}^{n_i}$ with rate R_i with distortion

$$\rho(C_i,\mu) \le \delta_{n_i}(R_i,\mu) + \epsilon.$$

Analogous to (12.6), for the product code $C = C_1 \times C_2$ we have

$$\begin{aligned} n\rho(C,\mu) &= n_1\rho(C_1,\mu) + n_2\rho(C_2,\mu) \\ &\le n_1\delta_{n_1}(R_1,\mu) + n_2\delta_{n_2}(R_2,\mu) + n\epsilon. \end{aligned}$$

The rate of the product code is no greater than R and hence the leftmost term above is bounded below by $n\delta_n(R,\mu)$. Dividing by n we have since ϵ is arbitrary that

$$\delta_n(R,\mu) \le \frac{n_1}{n}\delta_{n_1}(R_1,\mu) + \frac{n_2}{n}\delta_{n_2}(R_2,\mu).$$

Taking $n \to \infty$ we have using the lemma and the choice of n_i that

$$\delta(R,\mu) \le \lambda\delta(R_1,\mu) + (1-\lambda)\delta(R_2,\mu),$$

proving the claimed convexity. □

Corollary 12.3. *If μ is stationary, then $\delta(R,\mu)$ is an affine function of μ.*

Proof: From Lemma 12.2 we need only prove that

$$\delta(R,\lambda\mu_1 + (1-\lambda)\mu_2) \le \lambda\delta(R,\mu_1) + (1-\lambda)\delta(R,\mu_2).$$

From the same lemma we have that for any N

$$\delta_N(R + \frac{1}{N}, \lambda\mu_1 + (1-\lambda)\mu_2) \le \lambda\delta_N(R,\mu_1) + (1-\lambda)\delta_N(R,\mu_2)$$

For any $K \le N$ we have since $\delta_N(R,\mu)$ is nonincreasing in R that

$$\delta_N(R + \frac{1}{K}, \lambda\mu_1 + (1-\lambda)\mu_2) \le \lambda\delta_N(R,\mu_1) + (1-\lambda)\delta_N(R,\mu_2).$$

Taking the limit as $N \to \infty$ yields from Lemma 12.3 that

$$\delta(R + \frac{1}{K}, \mu) \le \lambda\delta(R,\mu_1) + (1-\lambda)\delta(R,\mu_2).$$

From Corollary 12.2, however, δ is continuous in R and the result follows by letting $K \to \infty$. □

The following lemma provides the principal tool necessary for relating the operational DRF of an AMS source with that of its stationary mean. It shows that the DRF of an AMS source is not changed by shifting or, equivalently, by redefining the time origin.

Lemma 12.4. *Let μ be an AMS source with a reference letter. Then for any integer i $\delta(R,\mu) = \delta(R,\mu T^{-i})$.*

Proof: Fix $\epsilon > 0$ and let C_N be a rate R block length N codebook for which $\rho(C_N,\mu) \le \delta(R,\mu) + \epsilon/2$. For $1 \le i \le N-1$ choose J large and define the block length $K = JN$ code $C_K(i)$ by

$$C_K(i) = a^{*(N-i)} \times \overset{J-2}{\underset{j=0}{\times}} C_N \times a^{*i},$$

where a^{*l} is an l-tuple containing all a^*'s. $C_K(i)$ can be considered to be a code consisting of the original code shifted by i time units and repeated many times, with some filler at the beginning and end. Except for the edges of the long product code, the effect on the source is to use the original code with a delay. The code has at most $(2^{NR})^{J-1} = 2^{KR}2^{-NR}$ words; the rate is no greater than R.

For any K-block x^K the distortion resulting from using $C_K^{(i)}$ is given by

$$K\rho_K(x^K, C_K(i)) \le (N-i)\rho_{N-i}(x^{N-i}, a^{*(N-i)}) + i\rho_i(x_{K-i}^i, a^{*i}). \quad (12.7)$$

Let $\{\hat{x}_n\}$ denote the encoded process using the block code $C_K(i)$. If n is a multiple of K, then

$$n\rho_n(x^n,\hat{x}^n) \le \sum_{k=0}^{\lfloor \frac{n}{K} \rfloor} ((N-i)\rho_{N-i}(x_{kK}^{N-i}, a^{*(N-i)}) + i\rho_i(x_{(k+1)K-i}^{i}, a^{*i})) + \sum_{k=0}^{\lfloor \frac{n}{K} \rfloor J-1} N\rho_N(x_{N-i+kN}^{N}, C_N).$$

If n is not a multiple of K we can further overbound the distortion by including the distortion contributed by enough future symbols to complete a K-block, that is,

$$\begin{aligned} n\rho_n(x^n,\hat{x}^n) &\le n\gamma_n(x,\hat{x}) \\ &= \sum_{k=0}^{\lfloor \frac{n}{K} \rfloor+1} \left((N-i)\rho_{N-i}(x_{kK}^{N-i}, a^{*(N-i)}) + i\rho_i(x_{(k+1)K-i}^{i}, a^{*i})\right) \\ &\quad + \sum_{k=0}^{(\lfloor \frac{n}{K} \rfloor+1)J-1} N\rho_N(x_{N-i+kN}^{N}, C_N). \end{aligned}$$

Thus

$$\begin{aligned} \rho_n(x^n,\hat{x}^n) \le{}& \frac{N-i}{K}\frac{1}{n/K} \sum_{k=0}^{\lfloor \frac{n}{K} \rfloor+1} \rho_{N-i}(X^{N-i}(T^{kK}x), a^{*(N-i)}) \\ &+ \frac{i}{K}\frac{1}{n/K} \sum_{k=0}^{\lfloor \frac{n}{K} \rfloor+1} \rho_i(X^i(T^{(k+1)K-i}x, a^{*i}) \\ &+ \frac{1}{n/N} \sum_{k=0}^{(\lfloor \frac{n}{K} \rfloor+1)J-1} \rho_N(X^N(T^{(N-i)+kN}x), C_N). \end{aligned}$$

Since μ is AMS these quantities all converge to invariant functions:

$$\begin{aligned} \lim_{n\to\infty} \rho_n(x^n,\hat{x}^n) \le{}& \frac{N-i}{K} \lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \rho_{N-i}(X^{N-i}(T^{kK}x), a^{*(N-i)}) \\ &+ \frac{i}{K} \lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \rho_i(X^i(T^{(k+1)K-i}x, a^{*i}) \\ &+ \lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \rho_N(X^N(T^{(N-i)+kN}x), C_N). \end{aligned}$$

We now apply Fatou's lemma, a change of variables, and Lemma 12.1 to obtain

$$\begin{aligned}
\delta(R,\mu T^{-i}) &\le \rho(C_K(i),\mu T^{-i}) \\
&\le \frac{N-i}{K}\limsup_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m} E_{\mu T^{-i}}\rho_{N-i}(X^{N-i}T^{kK},a^{*(N-i)}) \\
&\quad + \frac{i}{K}\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1} E_{\mu T^{-i}}\rho_i(X^iT^{(k+1)K-i},a^{*i}) \\
&\quad + E_{\mu T^{-i}}\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1}\rho_N(X^NT^{(N-i)+kN}),C_N). \\
&\le \frac{N-i}{K}\rho^* + \frac{i}{K}\rho^* + E_\mu\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{m-1}\rho_N(X^NT^{kN}C_N) \\
&\le \frac{N}{K}\rho^* + \rho(C_N,\mu).
\end{aligned}$$

Thus if J and hence K are chosen large enough to ensure that $N/K \le \epsilon/2$, then

$$\delta(R,\mu T^{-i}) \le \delta(R,\mu),$$

which proves that $\delta(R,\mu T^{-i}) \le \delta(R,\mu)$. The reverse implication is found in a similar manner: Let C_N be a codebook for μT^{-i} and construct a codebook $C_K(N-i)$ for use on μ. By arguments nearly identical to those above the reverse inequality is found and the proof completed. □

Corollary 12.4. *Let μ be an AMS source with a reference letter. Fix N and let $\overline{\mu}$ and $\overline{\mu}_N$ denote the stationary and N-stationary means. Then for $R > 0$*

$$\delta(R,\overline{\mu}) = \delta(R,\overline{\mu}_N T^{-i});\ i = 0,1,\ldots,N-1.$$

Proof: It follows from the previous lemma that the $\delta(R,\overline{\mu}_N T^{-i})$ are all equal and hence it follows from Lemma 12.2, Theorem 7.3.1 of [55] or Theorem 8.2 of [58], and Corollary 7.3.1 of [55] or Corollary 8.5 of [58] that

$$\delta(R,\overline{\mu}) \ge \frac{1}{N}\sum_{i=0}^{N-1}\delta(R,\overline{\mu}_N T^{-i}) = \delta(R,\overline{\mu}_N).$$

To prove the reverse inequality, take $\mu = \overline{\mu}_N$ in the previous lemma and construct the codes $C_K(i)$ as in the previous proof. Take the union code $C_K = \bigcup_{i=0}^{N-1} C_K(i)$ having block length K and rate at most $R + K^{-1}\log N$. We have from Lemma 12.1 and (12.7) that

$$\begin{aligned}
\rho(C_K,\overline{\mu}) &= \frac{1}{N}\sum_{i=0}^{N-1}\rho(C_K,\overline{\mu}_N T^{-i}) \\
&\le \frac{1}{N}\sum_{i=0}^{N-1}\rho(C_K(i),\overline{\mu}_N T^{-i}) \le \frac{N}{K}\rho^* + \rho(C_N,\overline{\mu}_N)
\end{aligned}$$

and hence as before

$$\delta(R + \frac{1}{JN}\log N, \overline{\mu}) \le \delta(R, \overline{\mu}_N).$$

From Corollary 12.1 $\delta(R,\overline{\mu})$ is continuous in R for $R > 0$ since $\overline{\mu}$ is stationary. Hence taking J large enough yields $\delta(R,\overline{\mu}) \le \delta(R,\overline{\mu}_N)$. This completes the proof since from the lemma $\delta(R,\overline{\mu}_N T^{-i}) = \delta(R,\overline{\mu}_N)$. □

We are now prepared to demonstrate the fundamental fact that the block source coding operational distortion-rate function for an AMS source with an additive fidelity criterion is the same as that of the stationary mean process. This will allow us to assume stationarity when proving the actual coding theorems.

Theorem 12.1. *If μ is an AMS source and $\{\rho_n\}$ an additive fidelity criterion with a reference letter, then for $R > 0$*

$$\delta(R,\mu) = \delta(R,\overline{\mu}).$$

Proof: We have from Corollaries 11.2.1 and 11.2.4 that

$$\delta(R,\overline{\mu}) \le \delta(R,\overline{\mu}_N) \le \delta_N(R,\overline{\mu}_N) = \delta_N(R,\mu).$$

Taking the infimum over N yields

$$\delta(R,\overline{\mu}) \le \delta(R,\mu).$$

Conversely, fix $\epsilon > 0$ let C_N be a block length N codebook for which $\rho(C_N,\overline{\mu}) \le \delta(R,\overline{\mu}) + \epsilon$. From Lemma 12.1, Corollary 12.1, and Lemma 12.4

$$\begin{aligned}
\delta(R,\overline{\mu}) + \epsilon \le \rho(C_N,\overline{\mu}) &= \frac{1}{N}\sum_{i=0}^{N-1} \rho(C_N,\overline{\mu}_N T^{-i}) \\
&\ge \frac{1}{N}\sum_{i=0}^{N-1} \delta_N(R,\overline{\mu}_N T^{-i}) = \frac{1}{N}\sum_{i=0}^{N-1} \delta_N(R,\mu T^{-i}) \\
&\ge \frac{1}{N}\sum_{i=0}^{N-1} \delta(R,\mu T^{-i}) = \delta(R,\mu),
\end{aligned}$$

which completes the proof since ϵ is arbitrary. □

Since the DRFs are the same for an AMS process and its stationary mean, this immediately yields the following corollary from Corollary 12.2:

Corollary 12.5. *If μ is AMS, then $\delta(R,\mu)$ is a convex function of R and hence a continuous function of R for $R > 0$.*

12.3 Block Source Code Mismatch

In this section the mismatch results of [66] for metric distortion measures are extended to additive distortion mesures that are a power of a metric, the class of fidelity criteria considered in Section 5.11. The formulation of operational distortion-rate functions for block codes in terms of the decoder or reproduction codebook alone can be combined with the process metric using the same distortion to quantify the difference in performance when a fixed code is applied on two distinct sources. The topic is of primary interest in the case where one designs a codebook to be optimal for one source, but then applies the codebook to actually code a different source. This can happen, for example, if the codebook is designed for a source by using an empirical distribution based on a training sequence, e.g., a clustering or learning algorithm is used on data to estimate the source statistics. The code so designed is then applied to the source itself, which results in a mismatch between the distribution used to design the code and the true, but unknown, underlying distribution. Intuitively, if the sources are close in some sense, then the performance of the code on the separate sources should also be close.

Suppose that $\{\rho_n\}$ is an additive fidelity criterion with per-symbol distortion that is a positive power of a metric, $\rho_1(x, y) = d(x, y)^p$, $p \geq 0$, as considered in Section 5.11. Fix a blocklength N, an N-dimensional reproduction codebook C, and two stationary sources with distributions μ_X and μ_Y. Recall from (5.40) that the rho-bar distortion between the two sources is

$$\overline{\rho}_N(\mu_{X^n}, \mu_{Y^n}) = \inf_{\pi \in \mathcal{P}(\mu_{X^N}, \mu_{Y^N})} E_\pi \rho_N(X^N, Y^N)$$

$$\overline{\rho}(\mu_X, \mu_Y) = \sup_N \frac{1}{N} \overline{\rho}_N(\mu_{X^N}, \mu_{Y^N}).$$

We assume the existence of a reference letter so that all of the expectations considered are finite. Fix N and suppose that π approximately yields $\overline{\rho}_N(\mu_{X^N}, \mu_{Y^N})$ in the sense that it has the correct marginals and for small $\epsilon > 0$

$$E_\pi \rho_N(X^N, Y^N) \leq \overline{\rho}_N(\mu_{X^N}, \mu_{Y^N}) + \epsilon.$$

For any x^N, y^N, z^N we have that

$$\rho_N(x^N, z^N) = \sum_{i=0}^{N-1} \rho_1(x_i, z_i) = \sum_{i=0}^{N-1} d(x_i, z_i)^p. \tag{12.8}$$

If d is a metric and $0 \le p \le 1$, then $\rho_N(x^N, z^N)$ is also a metric, in particular it satisfies the triangle inequality. Thus in this case we have that

$$\begin{aligned}\rho_N(C, \mu_X) &= E_{\mu_X}\left(\min_{z^N \in C} \rho_N(X^N, z^N)\right) = E_\pi\left(\min_{z^N \in C} \rho_N(X^N, z^N)\right) \\ &= E_\pi\left(\min_{z^N \in C}\left[\rho_N(X^N, Y^N) + \rho_N(Y^N, z^N)\right]\right) \\ &= E_\pi\left(\rho_N(X^N, Y^N)\right) + E_\pi\left(\min_{z^N \in C} \rho_N(Y^N, z^N)\right) \\ &\le \overline{\rho}_N(\mu_{X^N}, \mu_{Y^N}) + \epsilon + \rho_N(C, \mu_Y).\end{aligned}$$

Since ϵ is arbitrary,

$$\rho_N(C, \mu_X) \le \rho_N(C, \mu_Y) + \overline{\rho}_N(\mu_{X^N}, \mu_{Y^N}). \tag{12.9}$$

Reversing the roles of X and Y in (12.9) and using the fact that the process distortion is an upper bound for the normalized vector distortion implies the following bound on the mismatch in performance resulting from applying the same block code to different sources:

$$| N^{-1}\rho_N(C, \mu_X) - N^{-1}\rho_N(C, \mu_Y) | \le \overline{\rho}(\mu_X, \mu_Y) \tag{12.10}$$

Taking the infimum of both sides (12.10)over all codes C with $N^{-1}\log||C|| \le R$, normalizing the distortion, and using the fact that the process rho-bar distortion is an upper bound for all vector distortions yields

$$N^{-1}\delta_N(R, \mu_X) - N^{-1}\delta_N(R, \mu_Y) \le \overline{\rho}(\mu_X, \mu_Y).$$

Reversing the roles of X and Y yields

$$| N^{-1}\delta_N(R, \mu_X) - N^{-1}\delta_N(R, \mu_Y) | \le \overline{\rho}(\mu_X, \mu_Y) \tag{12.11}$$

which yields the conclusion that the N-th order operational distortion-rate functions are continuous functions of the source under the rho-bar distortion, which in this case of $0 \le p \le 1$ is the rho-bar distance. Since the sources are assumed stationary, the limits as $N \to \infty$ exist so that

$$| \delta(R, \mu_X) - \delta(R, \mu_Y) | \le \overline{\rho}(\mu_X, \mu_Y) \tag{12.12}$$

so that the operational block coding DRF is continuous with respect to rho-bar.

Now consider the case where $p > 1$ so that ρ_N does not satisfy a triangle inequality. Now, however,

$$\rho_N(x^N, z^N) = d_N^p(x^N, z^N) = \sum_{i=0}^{N-1} d(x_i, z_i)^p$$

is the pth power of a metric d_N, the ℓ_p norm on the vectors of the individual distortions. Analogous to the previous case consider

$$\begin{aligned}\rho_N(C,\mu_X) &= E_{\mu_X}\left(\min_{z^N\in C}\rho_N(X^N,z^N)\right)\\ &= E_\pi\left(\min_{z^N\in C}d_N(X^N,z^N)^p\right)\\ &= E_\pi\left(\min_{z^N\in C}[d_N(X^N,Y^N)+d_N(Y^N,z^N)]^p\right)\\ &= E_\pi\left([\min_{z^N\in C}(d_N(X^N,Y^N)+d_N(Y^N,z^N))]^p\right)\end{aligned}$$

since $f(x)=x^p$ is monotonically increasing for positive p and hence the minimum of the pth power of a quantity is the p power of the minimum. Continuing,

$$\rho_N(C,\mu_X)\le E_\pi\left([d_N(X^N,Y^N)+\min_{z^N\in C}(d_N(Y^N,z^N))]^p\right)$$

and hence application of Minkowski's inequality yields

$$\begin{aligned}\rho_N(C,\mu_X)^{1/p} &\le \left[E_\pi\left([d_N(X^N,Y^N)+\min_{z^N\in C}(d_N(Y^N,z^N))]^p\right)\right]^{1/p}\\ &\le \left[E_\pi\left(d_N(X^N,Y^N)^p\right)\right]^{1/p}+\left[E_\pi\left(\min_{z^N\in C}d_N(Y^N,z^N)^p\right)\right]^{1/p}\\ &\le [\overline{\rho}_N(\mu_{X^N},\mu_{Y^N})+\epsilon]^{1/p}+\rho_N(C,\mu_Y)^{1/p}.\end{aligned}$$

which since $\epsilon>0$ is arbitrary,

$$\rho_N(C,\mu_X)^{1/p}\le[\overline{\rho}_N(\mu_{X^N},\mu_{Y^N})]^{1/p}+\rho_N(C,\mu_Y)^{1/p}$$

which in a similar fashion to the previous case results in

$$\begin{aligned}|\,(N^{-1}\rho_N(C,\mu_X))^{1/p}-(N^{-1}\rho_N(C,\mu_Y))^{1/p}\,| &\le \overline{\rho}(\mu_X,\mu_Y)^{1/p}\\ |\,(N^{-1}\delta_N(R,\mu_X))^{1/p}-(N^{-1}\delta_N(R,\mu_Y))^{1/p}\,| &\le \overline{\rho}(\mu_X,\mu_Y)^{1/p}\\ |\,\delta(R,\mu_X)^{1/p}-\delta(R,\mu_Y)^{1/p}\,| &\le \overline{\rho}(\mu_X,\mu_Y)^{1/p}\end{aligned}$$

so that again the rho-bar distortion provides a bound on the performance mismatch of a single codebook used for different sources and the block source coding operational distortion-rate functions are continuous with respect to the rho-bar distortion.

This completes the proof of the following lemma.

Lemma 12.5. *Assume an additive fidelity criterion with per-letter distortion $\rho_1=d^p$, a positive power of a metric, μ_X and μ_Y two stationary process distributions, and C a reproduction codebook of length N.*

If $0 \le p \le 1$, *then*

$$| N^{-1}\rho_N(C,\mu_X) - N^{-1}\rho_N(C,\mu_Y) | \le \overline{\rho}(\mu_X,\mu_Y) \quad (12.13)$$
$$| N^{-1}\delta_N(R,\mu_X) - N^{-1}\delta_N(R,\mu_Y) | \le \overline{\rho}(\mu_X,\mu_Y) \quad (12.14)$$
$$| \delta(R,\mu_X) - \delta(R,\mu_Y) | \le \overline{\rho}(\mu_X,\mu_Y) \quad (12.15)$$

and if $1 \le p$, *then*

$$| (N^{-1}\rho_N(C,\mu_X))^{1/p} - (N^{-1}\rho_N(C,\mu_Y))^{1/p} | \le \overline{\rho}(\mu_X,\mu_Y)^{1/p} \quad (12.16)$$
$$| (N^{-1}\delta_N(R,\mu_X))^{1/p} - (N^{-1}\delta_N(R,\mu_Y))^{1/p} | \le \overline{\rho}(\mu_X,\mu_Y)^{1/p} \quad (12.17)$$
$$| \delta(R,\mu_X)^{1/p} - \delta(R,\mu_Y)^{1/p} | \le \overline{\rho}(\mu_X,\mu_Y)^{1/p} \quad (12.18)$$

Thus the block source coding operational distortion-rate functions $\delta_N(R,\mu)$ *and* $\delta(R,\mu)$ *are continuous functions of* μ *in the rho-bar distortion (and* $\overline{d}_p$ *distance).*

These mismatch results can be used to derive universal coding results for block source coding for certaom classes of sources. Universal codes are designed to provide nearly optimal coding for a collection of sources rather than for one specific source. The basic idea is to carve up the class using the rho-bar distortion (or the corresponding $\overline{d}_p$-distance) and to design a code for a specific representative of each subclass. If the members of a subclass are close in rho-bar, then the representative will work well for all sources in the subclass. The overall codebook is then formed as the union of the subclass codebooks. Provided the number of subclasses is small with respect to the block length, each subclass codebook can have nearly the full rate in bits per symbol and hence provide nearly optimal coding within the class. A minimum distortion rule encoder will find the best word within all of the classes. This approach to universal coding is detailed in [66, 132]. A variety of other approaches exist to this problem of source coding with uncertainty about the source, see for example [92, 198, 199].

12.4 Block Coding Stationary Sources

We showed in the previous section that when proving block source coding theorems for AMS sources, we could confine interest to stationary sources. In this section we show that in an important special case we can further confine interest to only those stationary sources that are ergodic by applying the ergodic decomposition. This will permit us to assume that sources are stationary and ergodic in the next section when the basic Shannon source coding theorem is proved and then extend the result to AMS sources which may not be ergodic.

As previously we assume that we have a stationary source $\{X_n\}$ with distribution μ and we assume that $\{\rho_n\}$ is an additive distortion measure and there exists a reference letter. For this section we now assume in addition that the alphabet A is itself a Polish space and that $\rho_1(r, y)$ is a continuous function of r for every $y \in \hat{A}$. If the underlying alphabet has a metric structure, then it is reasonable to assume that forcing input symbols to be very close in the underlying alphabet should force the distortion between either symbol and a fixed output to be close also. The following theorem is the ergodic decomposition of the block source coding operational distortion-rate function.

Theorem 12.2. *Suppose that μ is the distribution of a stationary source and that $\{\rho_n\}$ is an additive fidelity criterion with a reference letter. Assume also that $\rho_1(\cdot, y)$ is a continuous function for all y. Let $\{\mu_x\}$ denote the ergodic decomposition of μ. Then*

$$\delta(R, \mu) = \int d\mu(x)\delta(R, \mu_x),$$

that is, $\delta(R, \mu)$ is the average of the operational DRFs of its ergodic components.

Proof: Analogous to the ergodic decomposition of entropy rate of Theorem 3.3, we need to show that $\delta(R, \mu)$ satisfies the conditions of Theorem 8.9.1 of [55] or Theorem 8.5 of [58]. We have already seen (Corollary 12.3) that it is an affine function. We next see that it is upper semicontinuous. Since the alphabet is Polish, choose a distance $d_{\mathcal{G}}$ on the space of stationary processes having this alphabet with the property that $\mathcal{G}$ is constructed as in Section 8.2 of [55] or Section 9.8 of [58]. Pick an N large enough and a length N codebook C so that

$$\delta(R, \mu) \geq \delta_N(R, \mu) - \frac{\epsilon}{2} \geq \rho_N(C, \mu) - \epsilon.$$

$\rho_N(x^N, y)$ is by assumption a continuous function of x^N and hence so is $\rho_N(x^N, C) = \min_{y \in C} \rho(x^N, y)$. Since it is also nonnegative, we have from Lemma 8.2.4 of [55] or Lemma 9.3 of [58] that if $\mu_n \to \mu$ then

$$\limsup_{n \to \infty} E_{\mu_n} \rho_N(X^N, C) \leq E_{\mu} \rho_N(X^N, C).$$

The left hand side above is bounded below by

$$\limsup_{n \to \infty} \delta_N(R, \mu_n) \geq \limsup_{n \to \infty} \delta(R, \mu_n).$$

Thus since ϵ is arbitrary,

$$\limsup_{n \to \infty} \delta(R, \mu_n) \leq \delta(R, \mu)$$

and hence $\delta(R,\mu)$ upper semicontinuous in μ and hence also measurable. Since the process has a reference letter, $\delta(R,\mu_x)$ is integrable since

$$\delta(R,\mu_X) \le \delta_N(R,\mu_x) \le E_{\mu_x}\rho_1(X_0,a^*)$$

which is integrable if $\rho_1(x_0,a^*)$ is from the ergodic decomposition theorem. Thus Theorem 8.9.1 of [55] or Theorem 8.5 of [58] yields the desired result. □

The theorem was first proved by Kieffer [91] for bounded continuous additive distortion measures. The above extension removes the requirement that ρ_1 be bounded.

12.5 Block Coding AMS Ergodic Sources

We have seen that the block source coding operational DRF of an AMS source is given by that of its stationary mean. Hence we will be able to concentrate on stationary sources when proving the coding theorem.

Theorem 12.3. *Let μ be an AMS ergodic source with a standard alphabet and $\{\rho_n\}$ an additive distortion measure with a reference letter. Then*

$$\delta(R,\mu) = D(R,\overline{\mu}),$$

where $\overline{\mu}$ is the stationary mean of μ. If μ is stationary, then

$$\delta(R,\mu) = D(R,\mu).$$

Comment: Coupling the theorem with Lemma 12.5 shows that if the per-symbol distortion is a positive power of a metric, d^p , then the Shannon distortion rate function is a continuous function of the source distribution μ in terms of the corresponding rho-bar process distortion or the corresponding $\overline{d}_p$-distance. This has the same flavor of Corollary 6.2 showing that entropy was a continuous function of the d-bar distance, which assumed a mean Hamming distortion. The dual result to shows that the Shannon rate-distortion function is a continuous function of the source with respect to the rho-bar distortion.

Proof: From Theorem 12.1 $\delta(R,\mu) = \delta(R,\overline{\mu})$ and hence we will be done if we can prove that

$$\delta(R,\overline{\mu}) = D(R,\overline{\mu}).$$

This will follow if we can show that $\delta(R,\mu) = D(R,\mu)$ for any stationary ergodic source with a reference letter. Henceforth we assume that μ is stationary and ergodic.

The negative or converse half of the theorem follows from Corollary 9.1. As the specific case is simpler and short, a proof is included.

First suppose that we have a codebook C such that

$$\rho_N(C,\mu) = E_\mu \min_{y\in C} \rho_N(X^N, y) = \delta_N(R,\mu) + \epsilon.$$

If we let $\hat{X}_N$ denote the resulting reproduction random vector and let p^N denote the resulting joint distribution of the input/output pair, then since $\hat{X}^N$ has a finite alphabet, Lemma 7.20 implies that

$$I(X^N;\hat{X}^N) \le H(\hat{X}^N) \le NR$$

and hence $p^N \in \mathcal{R}_N(R,\mu^N)$ and hence

$$\delta_N(R,\mu) + \epsilon \ge E_{p^N}\rho_N(X^N;\hat{X}^N) \ge D_N(R,\mu).$$

Taking the limits as $N \to \infty$ proves the easy half of the theorem:

$$\delta(R,\mu) \ge D(R,\mu).$$

Recall that both operational DRF and the Shannon DRF are given by limits if the source is stationary.

The fundamental idea of Shannon's positive source coding theorem is this: for a fixed block size N, choose a code *at random* according to a distribution implied by the distortion-rate function. That is, perform 2^{NR} independent random selections of blocks of length N to form a codebook. This codebook is then used to encode the source using a minimum distortion mapping as above. We compute the average distortion over this double-random experiment (random codebook selection followed by use of the chosen code to encode the random source). We will find that if the code generation distribution is properly chosen, then this average will be no greater than $D(R,\mu) + \epsilon$. If the average over all randomly selected codes is no greater than $D(R,\mu) + \epsilon$, then there must be at least one code such that the average distortion over the source distribution for that one code is no greater than $D(R,\mu) + \epsilon$. This means that there exists at least one code with performance not much larger than $D(R,\mu)$. Unfortunately the proof only demonstrates the existence of such codes, it does not show how to construct them.

To find the distribution for generating the random codes we use the ergodic process definition of the Shannon distortion-rate function. From Theorem 9.1 (or Lemma 9.4) we can select a stationary and ergodic pair process with distribution p which has the source distribution μ as one coordinate and which has

$$E_p\rho(X_0,Y_0) = \frac{1}{N}E_{p^N}\rho_N(X^N,Y^N) \le D(R,\mu) + \epsilon \tag{12.19}$$

and which has

$$\overline{I}_p(X;Y) = I^*(X;Y) \le R \tag{12.20}$$

(and hence information densities converge in L^1 from Theorem 8.1). Denote the implied vector distributions for (X^N, Y^N), X^N, and Y^N by p^N, μ^N, and η^N, respectively.

For any N we can generate a codebook C at random according to η^N as described above. To be precise, consider the random codebook as a large random vector $C = (W_0, W_1, \cdots, W_M)$, where $M = \lfloor e^{N(R+\epsilon)} \rfloor$ (where natural logarithms are used in the definition of R), where W_0 is the fixed reference vector a^{*N} and where the remaining W_n are independent, and where the marginal distributions for the W_n are given by η^N. Thus the distribution for the randomly selected code can be expressed as

$$P_C = \overset{M}{\underset{i=1}{\times}} \eta^N.$$

This codebook is then used with the optimal encoder and we denote the resulting average distortion (over codebook generation and the source) by

$$\Delta_N = E\rho(C,\mu) = \int dP_C(\mathcal{W})\rho(\mathcal{W},\mu) \tag{12.21}$$

where

$$\rho(\mathcal{W},\mu) = \frac{1}{N}E\rho_N(X^N,\mathcal{W}) = \frac{1}{N}\int d\mu^N(x^N)\rho_N(x^N,\mathcal{W}),$$

and where

$$\rho_N(x^N,C) = \min_{\mathcal{Y}\in C}\rho_N(x^N,\mathcal{Y}).$$

Choose $\delta > 0$ and break up the integral over x into two pieces: one over a set $G_N = \{x : N^{-1}\rho_N(x^N, a^{*N}) \le \rho^* + \delta\}$ and the other over the complement of this set. Then

$$\Delta_N \le \int_{G_N^c} \frac{1}{N}\rho_N(x^N,a^{*N})\,d\mu^N(x^N) + \frac{1}{N}\int dP_C(\mathcal{W})\int_{G_N} d\mu^N(x^N)\rho_N(x^N,\mathcal{W}), \tag{12.22}$$

where we have used the fact that $\rho_N(x^N, m\mathcal{W}) \le \rho_N(x^N, a^{*N})$. Fubini's theorem implies that because

$$\int d\mu^N(x^N)\rho_N(x^N,a^{*N}) < \infty$$

and

$$\rho_N(x^N,\mathcal{W}) \le \rho_N(x^N,a^{*N}),$$

the limits of integration in the second integral of (12.22) can be interchanged to obtain the bound

$$\Delta_N \le \frac{1}{N}\int_{G_N^c} \rho_N(x^N, a^{*N})d\mu^N(x^N) + \frac{1}{N}\int_{G_N} d\mu^N(x^N)\int dP_C(\mathcal{W})\rho_N(x^N,\mathcal{W}) \quad (12.23)$$

The rightmost term in (12.23) can be bound above by observing that

$$\begin{aligned}
&\frac{1}{N}\int_{G_N} d\mu^N(x^N)[\int dP_C(\mathcal{W})\rho_N(x^N,\mathcal{W})] \\
&\quad = \frac{1}{N}\int_{G_N} d\mu^N(x^N)[\int_{C:\rho_N(x^N,C)\le N(D+\delta)} dP_C(\mathcal{W})\rho_N(x^N,\mathcal{W}) \\
&\qquad + \frac{1}{N}\int_{\mathcal{W}:\rho_N(x^N,\mathcal{W})>N(D+\delta)} dP_C(\mathcal{W})\rho_N(x^N,\mathcal{W})] \\
&\quad \le \int_{G_N} d\mu^N(x^N)[D+\delta+\frac{1}{N}(\rho^*+\delta)\int_{\mathcal{W}:\rho_N(x^N,\mathcal{W})>N(D+\delta)} dp_C(\mathcal{W})]
\end{aligned}$$

where we have used the fact that for $x \in G$ the maximum distortion is given by $\rho^* + \delta$. Define the probability

$$P(N^{-1}\rho_N(x^N, C) > D + \delta | x^N) = \int_{\mathcal{W}:\rho_N(x^N,\mathcal{W})>N(D+\delta)} dp_C(\mathcal{W})$$

and summarize the above bounds by

$$\begin{aligned}
\Delta_N \le D + \delta + &\\
&(\rho^*+\delta)\frac{1}{N}\int d\mu^N(x^N)P(N^{-1}\rho_N(x^N,C) > D+\delta|x^N) \\
&\qquad + \frac{1}{N}\int_{G_N^c} d\mu^N(x^N)\rho_N(x^N, a^{*N}). \quad (12.24)
\end{aligned}$$

The remainder of the proof is devoted to proving that the two integrals above go to 0 as $N \to \infty$ and hence

$$\limsup_{N\to\infty} \Delta_N \le D + \delta. \quad (12.25)$$

Consider first the integral

$$a_N = \frac{1}{N}\int_{G_N^c} d\mu^N(x^N)\rho_N(x^N, a^{*N}) = \int d\mu^N(x^N)1_{G_N^c}(x^N)\frac{1}{N}\rho_N(x^N, a^{*N}).$$

We shall see that this integral goes to zero as an easy application of the ergodic theorem. The integrand is dominated by $N^{-1}\rho_N(x^N, a^{*N})$ which

is uniformly integrable (Lemma 4.7.2 of [55] or Lemma 5.23 of [58]) and hence the integrand is itself uniformly integrable (Lemma 4.4.4 of [55] or Lemma 5.9 of [58]). Thus we can invoke the extended Fatou lemma (Lemma 4.4.5 of [55] or Lemma 5.10 of [58]) to conclude that

$$\begin{aligned}\limsup_{N\to\infty} a_N &\le \int d\mu^N(x^N) \limsup_{N\to\infty} \left(1_{G_N^c}(x^N)\frac{1}{N}\rho_N(x^N, a^{*N})\right) \\ &\le \int d\mu^N(x^N)(\limsup_{N\to\infty} 1_{G_N^c}(x^N))(\limsup_{N\to\infty} \frac{1}{N}\rho_N(x^N, a^{*N})).\end{aligned}$$

We have, however, that $\limsup_{N\to\infty} 1_{G_N^c}(x^N)$ is 0 unless $x^N \in G_N^c$ i.o. But this set has measure 0 since with μ_N probability 1, an x is produced so that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} \rho(x_i, a^*) = \rho^*$$

exists and hence with probability one one gets an x which can yield

$$N^{-1}\rho_N(x^N, a^{*N}) > \rho^* + \delta$$

at most for a finite number of N. Thus the above integral of the product of a function that is 0 a.e. with a dominated function must itself be 0 and hence

$$\limsup_{N\to\infty} a_N = 0. \tag{12.26}$$

We now consider the second integral in (12.24):

$$b_N = (\rho^* + \delta)\frac{1}{N}\int d\mu^N(x^N) P(N^{-1}\rho_N(x^N, C) > D + \delta | x^N).$$

Recall that $P(\rho_N(x^N, C) > D + \delta | x^N)$ is the probability that for a fixed input block x^N, a randomly selected code will result in a minimum distortion codeword larger than $D + \delta$. This is the probability that none of the M words (excluding the reference code word) selected independently at random according to to the distribution η^N lie within $D+\delta$ of the fixed input word x^N. This probability is bounded above by

$$P(\frac{1}{N}\rho_N(x^N, C) > D + \delta | x^N) \le [1 - \eta^N(\frac{1}{N}\rho_N(x^N, Y^N) \le D + \delta)]^M$$

where

$$\eta^N(\frac{1}{N}\rho_N(x^N, Y^N) \le D + \delta)) = \int_{y^N : \frac{1}{N}\rho_N(x^N, y^N) \le D+\delta} d\eta^N(y^N).$$

Now mutual information comes into the picture. The above probability can be bounded below by adding a condition:

$$\eta^N(\frac{1}{N}\rho_N(x^N, Y^N) \le D + \delta)$$
$$\ge \eta^N(\frac{1}{N}\rho_N(x^N, Y^N) \le D + \delta \text{ and } \frac{1}{N}i_N(x^N, Y^N) \le R + \delta),$$

where

$$\frac{1}{N}i_N(x^N, y^N) = \frac{1}{N}\ln f_N(x^N, y^N),$$

where

$$f_N(x^N, y^N) = \frac{dp^N(x^N, y^N)}{d(\mu^N \times \eta^N)(x^N, y^N)},$$

the Radon-Nikodym derivative of p^N with respect to the product measure $\mu^N \times \eta^N$. Thus we require both the distortion and the sample information be less than slightly more than their limiting value. Thus we have in the region of integration that

$$\frac{1}{N}i_N(x^N; y^N) = \frac{1}{N}\ln f_N(x^N, y^N) \le R + \delta$$

and hence

$$\eta_N(\rho_N(x^N, Y^N) \le D + \delta)$$
$$\ge \int_{y^N:\rho_N(x^N,y^N)\le D+\delta, f_N(x^N,y^N)\le e^{N(R+\delta)}} d\eta^N(y^N)$$
$$\ge e^{-N(R+\delta)} \int_{y^N:\rho_N(x^N,y^N)\le D+\delta, f_N(x^N,y^N)\le e^{N(R+\delta)}} d\eta^N(y^N) f_N(x^N, y^N)$$

which yields the bound

$$P(\frac{1}{N}\rho_N(x^N, C) > D + \delta | x^N)$$
$$\le [1 - \eta^N(\frac{1}{N}\rho_N(x^N, Y^N) \le D + \delta)]^M$$
$$\le [1 - e^{-N(R+\delta)} \int_{y^N:\frac{1}{N}\rho_N(x^N,y^N)\le D+\delta, \frac{1}{N}i_N(x^N,y^N)\le R+\delta} d\eta^N(y^N) f_N(x^N, y^N)]^M,$$

Applying the inequality

$$(1 - \alpha\beta)^M \le 1 - \beta + e^{-M\alpha}$$

for $\alpha, \beta \in [0, 1]$ yields

$$P(\frac{1}{N}\rho_N(x^N, C) > D + \delta | x^N) \leq$$
$$1 - \int_{y^N : \frac{1}{N}\rho_N(x^N,y^N) \leq D+\delta, \frac{1}{N} i_N(x^N,y^N) \leq R+\delta} d\eta^N(y^N) \times f_N(x^N, y^N) + e^{[-Me^{-N(R+\delta)}]}.$$

Averaging with respect to the distribution μ^N yields

$$\frac{b_N}{\rho^* + \delta} = \int d\mu^N(x^N) P(\rho_N(x^N, C) > D + \delta | x^N)$$
$$\leq \int d\mu^N(x^N) \left(1 - \int_{y^N : \rho_N(x^N,y^N) \leq N(D+\delta), \frac{1}{N} i_N(x^N,y^N) \leq R+\delta} d\eta^N(y^N) \times f_N(x^N, y^N) + e^{-Me^{-N(R+\delta)}}\right)$$
$$= 1 - \int_{y^N : \frac{1}{N}\rho_N(x^N,y^N) \leq D+\delta, \frac{1}{N} i_N(x^N,y^N) \leq R+\delta} d(\mu^N \times \eta^N)(x^N, y^N) \times f_N(x^N, y^N) + e^{-Me^{-N(R+\delta)}}$$
$$= 1 + e^{-Me^{-N(R+\delta)}} - \int_{y^N : \frac{1}{N}\rho_N(x^N,y^N) \leq D+\delta, \frac{1}{N} i_N(x^N,y^N) \leq R+\delta} dp^N(x^N, y^N)$$
$$= 1 + e^{-Me^{-N(R+\delta)}}$$
$$- p^N(y^N : \frac{1}{N}\rho_N(x^N, y^N) \leq D + \delta, \frac{1}{N} i_N(x^N, y^N) \leq R + \delta). \quad (12.27)$$

Since M is bounded below by $e^{N(R+\epsilon)} - 1$, the exponential term is bounded above by

$$e^{[-e^{(N(R+\epsilon)}e^{-N(R+\delta)} + e^{-N(R+\delta)}]} = e^{[-e^{N(\epsilon-\delta)} + e^{-N(R+\delta)}]}.$$

If $\epsilon > \delta$, this term goes to 0 as $N \to \infty$.

The probability term in (12.27) goes to 1 from the mean ergodic theorem applied to ρ_1 and the mean ergodic theorem for information density since mean convergence (or the almost everywhere convergence proved elsewhere) implies convergence in probability. This implies that

$$\limsup_{n \to \infty} b_N = 0$$

which with (12.26) gives (12.25). Choosing an N so large that $\Delta_N \leq \delta$, we have proved that there exists a block code C with average distortion less than $D(R, \mu) + \delta$ and rate less than $R + \epsilon$ and hence

$$\delta(R + \epsilon, \mu) \leq D(R, \mu) + \delta. \quad (12.28)$$

Since ϵ and δ can be chosen as small as desired and since $D(R, \mu)$ is a continuous function of R (Lemma 9.1), the theorem is proved. □

The source coding theorem is originally due to Shannon [162] [163], who proved it for discrete IID sources. It was extended to stationary and ergodic discrete alphabet sources and Gaussian sources by Gallager [47] and to stationary and ergodic sources with abstract alphabets by Berger [10] [11], but an error in the information density convergence result of Perez [148] (see Kieffer [89]) left a gap in the proof, which was subsequently repaired by Dunham [36]. The result was extended to nonergodic stationary sources and metric distortion measures and Polish alphabets by Gray and Davisson [59] and to AMS ergodic processes by Gray and Saadat [70]. The method used here of using a stationary and ergodic measure to construct the block codes and thereby avoid the block ergodic decomposition of Nedoma [129] used by Gallager [47] and Berger [11] was suggested by Pursley and Davisson [29] and developed in detail by Gray and Saadat [70].

12.6 Subadditive Fidelity Criteria

In this section we generalize the block source coding theorem for stationary sources to subadditive fidelity criteria. Several of the interim results derived previously are no longer appropriate, but we describe those that are still valid in the course of the proof of the main result. Most importantly, we now consider only stationary and not AMS sources. The result can be extended to AMS sources in the two-sided case, but it is not known for the one-sided case. Source coding theorems for subadditive fidelity criteria were first developed by Mackenthun and Pursley [111].

Theorem 12.4. *Let μ denote a stationary and ergodic distribution of a source $\{X_n\}$ and let $\{\rho_n\}$ be a subadditive fidelity criterion with a reference letter, i.e., there is an $a^* \in \hat{A}$ such that*

$$E\rho_1(X_0, a^*) = \rho^* < \infty.$$

Then the operational DRF for the class of block codes of rate less than R is given by the Shannon distortion-rate function $D(R, \mu)$.

Proof: Suppose that we have a block code of length N, e.g., a block encoder $\alpha : A^N \to B^K$ and a block decoder $\beta : B^K \to \hat{A}^N$. Since the source is stationary, the induced input/output distribution is then N-stationary and the performance resulting from using this code on a source μ is

$$\Delta_N = E_p \rho_\infty = \frac{1}{N} E_p \rho_N(X^N, \hat{X}^N),$$

where $\{\hat{X}^N\}$ is the resulting reproduction process. Let $\delta_N(R, \mu)$ denote the infimum over all codes of length N of the performance using such

codes and let $\delta(R,\mu)$ denote the infimum of δ_N over all N, that is, the operational distortion rate function. We do not assume a codebook/minimum distortion structure because the distortion is now effectively context dependent and it is not obvious that the best codes will have this form. Assume that given an $\epsilon > 0$ we have chosen for each N a length N code such that

$$\delta_N(R,\mu) \geq \Delta_N - \epsilon.$$

As previously we assume that

$$\frac{K \log ||B||}{N} \leq R,$$

where the constraint R is the rate of the code. As in the proof of the converse coding theorem for an additive distortion measure, we have that for the resulting process $I(X^N;\hat{X}^N) \leq RN$ and hence

$$\Delta_N \geq D_N(R,\mu).$$

From Lemma 9.2 we can take the infimum over all N to find that

$$\delta(R,\mu) = \inf_N \delta_N(R,\mu) \geq \inf_N D_N(R,\mu) - \epsilon = D(R,\mu) - \epsilon.$$

Since ϵ is arbitrary, $\delta(R,\mu) \leq D(R,\mu)$, proving the converse theorem.

To prove the positive coding theorem we proceed in an analogous manner to the proof for the additive case, except that we use Lemma 9.4 instead of Theorem 9.1. First pick an N large enough so that

$$D_N(R,\mu) \leq D(R,\mu) + \frac{\delta}{2}$$

and then select a $p^N \in \mathcal{R}_N(R,\mu^N)$ such that

$$E_{p^N} \frac{1}{N} \rho_N(X^N, Y^N) \leq D_N(R,\mu) + \frac{\delta}{2} \leq D(R,\mu) + \delta.$$

Construct as in Lemma 9.4 a stationary and ergodic process p which will have (10.6.4) and (10.6.5) satisfied (the right Nth order distortion and information). This step taken, the proof proceeds exactly as in the additive case since the reference vector yields the bound

$$\frac{1}{N} \rho_N(x^N, a^{*N}) \leq \frac{1}{N} \sum_{i=0}^{N-1} \rho_1(x_i, a^*),$$

which converges, and since $N^{-1}\rho_N(x^N, y^N)$ converges as $N \to \infty$ with p probability one from the subadditive ergodic theorem. Thus the exis-

tence of a code satisfying (12.28) can be demonstrated (which uses the minimum distortion encoder) and this implies the result since $D(R,\mu)$ is a continuous function of R (Lemma 9.1). □

12.7 Asynchronous Block Codes

The block codes considered so far all assume block synchronous communication, that is, that the decoder knows where the blocks begin and hence can deduce the correct words in the codebook from the index represented by the channel block. In this section we show that we can construct asynchronous block codes with little loss in performance or rate; that is, we can construct a block code so that a decoder can uniquely determine how the channel data are parsed and hence deduce the correct decoding sequence. This result will play an important role in the development in the next section of sliding-block coding theorems. The basic approach is that taken in the development of asynchronous and sliding-block almost lossless codes in Section 6.5.

Given a source μ let $\delta_{\text{async}}(R,\mu)$ denote the operational distortion rate function for block codes with the added constraint that the decoder be able to synchronize, that is, correctly parse the channel codewords. Obviously

$$\delta_{\text{async}}(R,\mu) \geq \delta(R,\mu)$$

since we have added a constraint. The goal of this section is to prove the following result:

Theorem 12.5. *Given an AMS source with an additive fidelity criterion and a reference letter,*

$$\delta_{async}(R,\mu) = \delta(R,\mu),$$

that is, the operational DRF for asynchronous codes is the same as that for ordinary codes.

Proof: A simple way of constructing a synchronized block code is to use a prefix code: Every codeword begins with a short prefix or *source synchronization word* or, simply, sync word, that is not allowed to appear anywhere else within a word or as any part of an overlap of the prefix and a piece of the word. The decoder then need only locate the prefix in order to decode the block begun by the prefix. The insertion of the sync word causes a reduction in the available number of codewords and hence a loss in rate, but ideally this loss can be made negligible if properly done. We construct a code in this fashion by finding a good codebook of slightly smaller rate and then indexing it by channel K-tuples with this prefix property.

Suppose that our channel has a rate constraint R, that is, if source N-tuples are mapped into channel K-tuples then

$$\frac{K \log ||B||}{N} \leq R,$$

where B is the channel alphabet. We assume that the constraint is achievable on the channel in the sense that we can choose N and K so that the physical stationarity requirement is met (N source time units corresponds to K channel time units) and such that

$$||B||^K \approx e^{NR}, \tag{12.29}$$

at least for large N.

If K is to be the block length of the channel code words, let δ be small and define $k(K) = \lfloor \delta K \rfloor + 1$ and consider channel codewords which have a prefix of $k(K)$ occurrences of a single channel letter, say b, followed by a sequence of $K - k(K)$ channel letters which have the following constraint: no $k(K)$-tuple beginning after the first symbol can be $b^{k(K)}$. We permit b's to occur at the end of a K-tuple so that a $k(K)$-tuple of b's may occur in the overlap of the end of a codeword and the new prefix since this causes no confusion, e.g., if we see an elongated sequence of b's, the actual code information starts at the right edge. Let $M(K)$ denote the number of distinct channel K-tuples of this form. Since $M(K)$ is the number of distinct reproduction codewords that can be indexed by channel codewords, the codebooks will be constrained to have rate

$$R_K = \frac{\ln M(K)}{N}.$$

We now study the behavior of R_K as K gets large. There are a total of $||B||^{K-k(K)}$ K-tuples having the given prefix. Of these, no more than $(K - k(K))||B||^{K-2k(K)}$ have the sync sequence appearing somewhere within the word (there are fewer than $K - k(K)$ possible locations for the sync word and for each location the remaining $K - 2k(K)$ symbols can be anything). Lastly, we must also eliminate those words for which the first i symbols are b for $i = 1, 2, \ldots, k(K) - 1$ since this will cause confusion about the right edge of the sync sequence. These terms contribute

$$\sum_{i=1}^{k(K)-1} ||B||^{K-k(K)-i}$$

bad words. Using the geometric progression formula to sum the above series we have that it is bounded above by

$$\frac{||B||^{K-k(K)-1}}{1 - 1/||B||}.$$

Thus the total number of available channel vectors is at least

$$M(K) \ge ||B||^{K-k(K)} - (K-k(K))||B||^{K-2k(K)} - \frac{||B||^{K-k(K)-1}}{1-1/||B||}.$$

Thus

$$\begin{aligned} R_K &= \frac{1}{N}\ln||B||^{K-k(K)} + \frac{1}{N}\ln\left(1-(K-k(K))||B||^{-k(K)} - \frac{1}{||B||-1}\right) \\ &= \frac{K-k(K)}{N}\ln||B|| + \frac{1}{N}\ln\left(\frac{||B||-2}{||B||-1} - (K-k(K))||B||^{-k(K)}\right) \\ &\ge (1-\delta)R + o(N), \end{aligned}$$

where $o(N)$ is a term that goes to 0 as N (and hence K) goes to infinity. Thus given a channel with rate constraint R and given $\epsilon > 0$, we can construct for N sufficiently large a collection of approximately $e^{N(R-\epsilon)}$ channel K-tuples (where $K \approx NR$) which are synchronizable, that is, satisfy the prefix condition.

We are now ready to construct the desired code. Fix $\delta > 0$ and then choose $\epsilon > 0$ small enough to ensure that

$$\delta(R(1-\epsilon),\mu) \le \delta(R,\mu) + \frac{\delta}{3}$$

(which we can do since $\delta(R,\mu)$ is continuous in R). Then choose an N large enough to give a prefix channel code as above and to yield a rate $R-\epsilon$ codebook C so that

$$\begin{aligned} \rho_N(C,\mu) &\le \delta_N(R-\epsilon,\mu) + \frac{\delta}{3} \\ &\le \delta(R-\epsilon,\mu) + \frac{2\delta}{3} \le \delta(R,\mu) + \delta. \end{aligned} \tag{12.30}$$

The resulting code proves the theorem. □

12.8 Sliding-Block Source Codes

We now turn to sliding-block codes. For simplicity we consider codes which map blocks into single symbols. For example, a sliding-block encoder will be a mapping $f : A^N \to B$ and the decoder will be a mapping $g : B^K \to \hat{A}$. In the case of one-sided processes, for example, the channel sequence would be given by

$$U_n = f(X_n^N)$$

and the reproduction sequence by

$$\hat{X}_n = g(U_n^L).$$

When the processes are two-sided, it is more common to use memory as well as delay. This is often done by having an encoder mapping $f : A^{2N+1} \to B$, a decoder $g : B^{2L+1} \to \hat{A}$, and the channel and reproduction sequences being defined by

$$U_n = f(X_{-N}, \cdots, X_0, \cdots, X_N),$$
$$\hat{X}_n = g(U_{-L}, \cdots, U_0, \cdots, U_N).$$

We emphasize the two-sided case.

The final output can be viewed as a sliding-block coding of the input:

$$\begin{aligned}\hat{X}_n &= g(f(X_{n-L-N}, \cdots, X_{n-L+N}), \cdots, f(X_{n+L-N}, \cdots, X_{n+L+N})) \\ &= gf(X_{n-(N+L)}, \cdots, X_{n+(N+L)}),\end{aligned}$$

where we use gf to denote the overall coding, that is, the cascade of g and f. Note that the delay and memory of the overall code are the sums of those for the encoder and decoder. The overall window length is $2(N + L) + 1$

Since one channel symbol is sent for every source symbol, the rate of such a code is given simply by $R = \log ||B||$ bits per source symbol. The obvious problem with this restriction is that we are limited to rates which are logarithms of integers, e.g., we cannot get fractional rates. As previously discussed, however, we could get fractional rates by appropriate redefinition of the alphabets (or, equivalently, of the shifts on the corresponding sequence spaces). For example, regardless of the code window lengths involved, if we shift l source symbols to produce a new group of k channel symbols (to yield an (l, k)-stationary encoder) and then shift a group of k channel symbols to produce a new group of k source symbols, then the rate is

$$R = \frac{k}{l} \log ||B||$$

bits or nats per source symbol and the overall code fg is l-stationary. The added notation to make this explicit is significant and the generalization is straightforward; hence we will stick to the simpler case.

We can define the sliding-block operational DRF for a source and channel in the natural way. Suppose that we have an encoder f and a decoder g. Define the resulting performance by

$$\rho(fg, \mu) = E_{\mu fg}\rho_\infty,$$

where μfg is the input/output hookup of the source μ connected to the deterministic channel fg and where ρ_∞ is the sequence distortion.

Define

$$\delta_{\mathrm{SBC}}(R,\mu) = \inf_{f,g} \rho(fg,\mu) = \Delta(\mu,\mathcal{E},\nu,\mathcal{D}), \tag{12.31}$$

where $\mathcal{E}$ is the class of all finite-length sliding-block encoders and $\mathcal{D}$ is the collection of all finite-length sliding-block decoders. The rate constraint R is determined by the channel.

Assume as usual that μ is AMS with stationary mean $\overline{\mu}$. Since the cascade of stationary channels fg is itself stationary (Lemma 2.10), we have from Lemma 2.2 that μfg is AMS with stationary mean $\overline{\mu}fg$. This implies from (5.12) that for any sliding-block codes f and g

$$E_{\mu fg}\rho_\infty = E_{\overline{\mu}fg}\rho_\infty$$

and hence

$$\delta_{\mathrm{SBC}}(R,\mu) = \delta_{\mathrm{SBC}}(R,\overline{\mu}).$$

A fact we now formalize as a lemma.

Lemma 12.6. *Suppose that μ is an AMS source with stationary mean $\overline{\mu}$ and let $\{\rho_n\}$ be an additive fidelity criterion. Let $\delta_{\mathrm{SBC}}(R,\mu)$ denote the sliding-block coding operational distortion-rate function for the source and a channel with rate constraint R. Then*

$$\delta_{\mathrm{SBC}}(R,\mu) = \delta_{\mathrm{SBC}}(R,\overline{\mu}).$$

The lemma permits us to concentrate on stationary sources when quantifying the optimal performance of sliding-block codes.

The principal result of this section is the following:

Theorem 12.6. *Given an AMS and ergodic source μ and an additive fidelity criterion with a reference letter,*

$$\delta_{\mathrm{SBC}}(R,\mu) = \delta(R,\mu),$$

that is, the class of sliding-block codes is capable of exactly the same performance as the class of block codes. If the source is only AMS and not ergodic, then

$$\delta_{\mathrm{SBC}}(R,\mu) \geq \delta(R,\mu), \tag{12.32}$$

Proof: The proof of (12.32) follows that of Shields and Neuhoff [167] for the finite alphabet case, except that their proof was for ergodic sources and coded only typical input sequences. Their goal was different because they measured the rate of a sliding-block code by the entropy rate of its output, effectively assuming that further almost-noiseless coding was to be used. Because we consider a fixed channel and measure the rate in the usual way as a coding rate, this problem does not arise here. From the previous lemma we need only prove the result for stationary sources and hence we henceforth assume that μ is stationary. We first prove

that sliding-block codes can perform no better than block codes, that is, (12.32) holds. Fix $\delta > 0$ and suppose that $f : A^{2N+1} \to B$ and $g : B^{2L+1} \to \hat{A}$ are finite-length sliding-block codes for which

$$\rho(fg, \mu) \le \delta_{\text{SBC}}(R, \mu) + \delta.$$

This yields a cascade sliding-block code $fg : A^{2(N+L)+1} \to \hat{A}$ which we use to construct a block codebook. Choose K large (to be specified later). Observe an input sequence x^n of length $n = 2(N + L) + 1 + K$ and map it into a reproduction sequence $\hat{x}^n$ as follows: Set the first and last $(N + L)$ symbols to the reference letter a^*, that is, $x_0^{N+L} = x_{n-N-L}^{N+L} = a^{*(N+L)}$. Complete the remaining reproduction symbols by sliding-block coding the source word using the given codes, that is,

$$\hat{x}_i = fg(x_{i-(N+L)}^{2(N+L)+1}); \; i = N + L + 1, \cdots, K + N + L.$$

Thus the long block code is obtained by sliding-block coding, except at the edges where the sliding-block code is not permitted to look at previous or future source symbols and hence are filled with a reference symbol. Call the resulting codebook C. The rate of the block code is less than $R = \log ||B||$ because n channel symbols are used to produce a reproduction word of length n and hence the codebook can have no more that $||B||^n$ possible vectors. Thus the rate is $\log ||B||$ since the codebook is used to encode a source n-tuple. Using this codebook with a minimum distortion rule can do no worse (except at the edges) than if the original sliding-block code had been used and therefore if $\hat{X}_i$ is the reproduction process produced by the block code and Y_i that produced by the sliding-block code, we have (invoking stationarity) that

$$\begin{aligned} &n\rho(C, \mu) \\ &\le E\Big(\sum_{i=0}^{N+L-1} \rho(X_i, a^*)\Big) + E\Big(\sum_{i=N+L}^{K+N+L} \rho(X_i, Y_i)\Big) + E\Big(\sum_{i=K+N+L+1}^{K+2(L+N)} \rho(X_i, a^*)\Big) \\ &\le 2(N + L)\rho^* + K(\delta_{\text{SBC}}(R, \mu) + \delta) \end{aligned}$$

and hence

$$\delta(R, \mu) \le \frac{2(N + L)}{2(N + L) + K}\rho^* + \frac{K}{2(N + L) + K}(\delta_{\text{SBC}}(R, \mu) + \delta).$$

By choosing δ small enough and K large enough we can make make the right hand side arbitrarily close to $\delta_{\text{SBC}}(R, \mu)$, which proves (12.32).

We now proceed to prove the converse inequality,

$$\delta(R, \mu) \ge \delta_{\text{SBC}}(R, \mu), \tag{12.33}$$

which involves a bit more work.

Before carefully tackling the proof, we note the general idea and an "almost proof" that unfortunately does not quite work, but which may provide some insight. Suppose that we take a very good block code, e.g., a block code C of block length N such that

$$\rho(C, \mu) \le \delta(R, \mu) + \delta$$

for a fixed $\delta > 0$. We now wish to form a sliding-block code for the same channel with approximately the same performance. Since a sliding-block code is just a stationary code (at least if we permit an infinite window length), the goal can be viewed as "stationarizing" the nonstationary block code. One approach would be the analogy of the SBM channel. Since a block code can be viewed as a deterministic block memoryless channel, we could make it stationary by inserting occasional random spacing between long sequences of blocks. Ideally this would then imply the existence of a sliding-block code from the properties of SBM channels. The problem is that the SBM channel so constructed would no longer be a deterministic coding of the input since it would require the additional input of a random punctuation sequence. Nor could one use a random coding argument to claim that there must be a specific (nonrandom) punctuation sequence which could be used to construct a code since the deterministic encoder thus constructed would not be a stationary function of the input sequence, that is, it is only stationary if both the source and punctuation sequences are shifted together. Thus we are forced to obtain the punctuation sequence from the source input itself in order to get a stationary mapping. The original proofs for this result [65, 67] used a strong form of the Rohlin-Kakutani theorem of Section 2.22 given by Shields [164]. The Rohlin-Kakutani theorem demonstrates the existence of a punctuation sequence with the property that the punctuation sequence is very nearly independent of the source. Lemma 2.12 is a slightly weaker result than the form considered by Shields.

The code construction described above can therefore be approximated by using a coding of the source instead of an independent process. Shields and Neuhoff [167] provided a simpler proof of a result equivalent to the Rohlin-Kakutani theorem and provided such a construction for finite alphabet sources. Davisson and Gray [28] provided an alternative heuristic development of a similar construction. We here adopt a somewhat different tack in order to avoid some of the problems arising in extending these approaches to general alphabet sources and to nonergodic sources. The principal difference is that we do not try to prove or use any approximate independence between source and the punctuation process derived from the source (which is code dependent in the case of continuous alphabets). Instead we take a good block code and first produce a much longer block code that is insensitive to shifts or starting positions using the same construction used to relate block cod-

ing performance of AMS processes to that of their stationary mean. This modified block code is then made into a sliding-block code using a punctuation sequence derived from the source. Because the resulting block code is little affected by starting time, the only important property is that most of the time the block code is actually in use. Independence of the punctuation sequence and the source is no longer required. The approach is most similar to that of Davisson and Gray [28], but the actual construction differs in the details. An alternative construction may be found in Kieffer [93].

Given $\delta > 0$ and $\epsilon > 0$, choose for large enough N an asynchronous block code C of block length N such that

$$\frac{1}{N} \log ||C|| \le R - 2\epsilon$$

and

$$\rho(C, \mu) \le \delta(R, \mu) + \delta. \tag{12.34}$$

The continuity of the block operational distortion-rate function and the theorem for asynchronous block source coding ensure that we can do this. Next we construct a longer block code that is more robust against shifts. For $i = 0, 1, \ldots, N-1$ construct the codes $C_K(i)$ having length $K = JN$ as in the proof of Lemma 12.4. These codebooks look like $J-1$ repetitions of the codebook C starting from time i with the leftover symbols at the beginning and end being filled by the reference letter. We then form the union code $C_K = \bigcup_i C_K(i)$ as in the proof of Corollary 12.4 which has all the shifted versions. This code has rate no greater than $R - 2\epsilon + (JN)^{-1} \log N$. We assume that J is large enough to ensure that

$$\frac{1}{JN} \log N \le \epsilon \tag{12.35}$$

so that the rate is no greater than $R - \epsilon$ and that

$$\frac{3}{J} \rho^* \le \delta. \tag{12.36}$$

We now construct a sliding-block encoder f and decoder g from the given block code. From Corollary 2.1 we can construct a finite length sliding-block code of $\{X_n\}$ to produce a two-sided (NJ, γ)-random punctuation sequence $\{Z_n\}$. From the lemma $P(Z_0 = 2) \le \gamma$ and hence by the continuity of integration (Corollary 4.4.2 of [55] or Corollary 5.3 in [58]) we can choose γ small enough to ensure that

$$\int_{x:Z_0(x)=2} \rho(X_0, a^*) \le \delta. \tag{12.37}$$

Recall that the punctuation sequence usually produces 0's followed by $NJ - 1$ 1's with occasional 2's interspersed to make things stationary. The sliding-block encoder f begins with time 0 and scans backward NJ time units to find the first 0 in the punctuation sequence. If there is no such 0, then put out an arbitrary channel symbol b. If there is such a 0, then the block codebook C_K is applied to the input K-tuple x_{-n}^K to produce the minimum distortion codeword

$$u^K = \min_{y \in C_K}{}^{-1} \rho_K(x_{-n}^K, y)$$

and the appropriate channel symbol, u_n, produced by the channel. The sliding-block encoder thus has length at most $2NJ + 1$.

The decoder sliding-block code g scans left N symbols to see if it finds a codebook sync sequence (remember the codebook is asynchronous and begins with a unique prefix or sync sequence). If it does not find one, it produces a reference letter. (In this case it is not in the middle of a code word.) If it does find one starting in position $-n$, then it produces the corresponding length N codeword from C and then puts out the reproduction symbol in position n. Note that the decoder sliding-block code has a finite window length of at most $2N + 1$.

We now evaluate the average distortion resulting from use of this sliding-block code. As a first step we mimic the proof of Lemma 9.4 up to the assumption of mutual independence of the source and the punctuation process (which is not the case here) to infer that if a long source sequence of length n yields the punctuation sequence z, then

$$\rho_n(x^n, \hat{x}^n) = \sum_{i \in J_2^n(z)} \rho(x_i, a^*) + \sum_{i \in J_0^n(z)} \rho_{NJ}(x_i^{NJ}, \hat{x}_i^{NJ}),$$

where $J_2^n(z)$ is the collection of all i for which $z_i = 2$ and hence z_i is not in an NJ-cell (so that filler is being sent) and $J_0^n(z)$ is the collection of all i for which z_i is 0 and hence begins an NJ-cell and hence an NJ length codeword. Each one of these length NJ codewords contains at most N reference letters at the beginning and N reference letters at the end the end and in the middle it contains all shifts of sequences of length N codewords from C. Thus for any $i \in J_0^n(z)$, we can write that

$$\rho_{NJ}(x_i^{NJ}, \hat{x}_i^{NJ}) \le \rho_N(x_i^N, a^{*N}) + \rho_N(x_{i+NJ-N}^N, a^{*N}) + \sum_{j=\lfloor \frac{i}{N} \rfloor}^{\lfloor \frac{i}{N} \rfloor + JN - 1} \rho_N(x_j^N, C).$$

This yields the bound

$$\frac{1}{n}\rho_n(x^n,\hat{x}^n) \le \frac{1}{n}\sum_{i\in J_2^n(z)}\rho(x_i,a^*)+$$

$$\frac{1}{n}\sum_{i\in J_0^n(z)}\left(\rho_N(x_i^N,a^{*N})+\rho_N(x_{i+NJ-N}^N,a^{*N})\right)+\frac{1}{n}\sum_{j=0}^{\lfloor\frac{n}{N}\rfloor}\rho_N(x_{jN}^N,C)$$

$$=\frac{1}{n}\sum_{i=0}^{n-1}1_2(z_i)\rho(x_i,a^*)+$$

$$\frac{1}{n}\sum_{i=0}^{n-1}1_0(z_i)\left(\rho_N(x_i^N,a^{*N})+\rho_N(x_{i+NJ-N}^N,a^{*N})\right)+\frac{1}{n}\sum_{j=0}^{\lfloor\frac{n}{N}\rfloor}\rho_N(x_{jN}^N,C),$$

where as usual the indicator function $1_a(z_i)$ is 1 if $z_i = a$ and 0 otherwise. Taking expectations above we have that

$$E\left(\frac{1}{n}\rho_n(X^n,\hat{X}^n)\right) \le \frac{1}{n}\sum_{i=0}^{n-1}E\left[1_2(Z_i)\rho(X_i,a^*)\right]$$

$$+\frac{1}{n}\sum_{i=0}^{n-1}E\left[1_0(Z_i)\left(\rho_N(X_i^N,a^{*N})+\rho_N(X_{i+NJ-N}^N,a^{*N})\right)\right]$$

$$+\frac{1}{n}\sum_{j=0}^{\lfloor\frac{n}{N}\rfloor}E\left[\rho_N(X_{jN}^N,C)\right].$$

Invoke stationarity to write

$$E(\frac{1}{n}\rho_n(X^n,\hat{X}^n)) \le E(1_2(Z_0)\rho(X_0,a^*))+$$

$$\frac{1}{NJ}E(1_0(Z_0)\rho_{2N+1}(X^{2N+1},a^{*(2N+1)}))+\frac{1}{N}\rho_N(X^N,C).$$

The first term is bounded above by δ from (12.37). The middle term can be bounded above using (12.36) by

$$\frac{1}{JN}E(1_0(Z_0)\rho_{2N+1}(X^{2N+1},a^{*(2N+1)}) \le \frac{1}{JN}E\rho_{2N+1}(X^{2N+1},a^{*(2N+1)})$$

$$=\frac{1}{JN}(2N+1)\rho^* \le (\frac{2}{J}+1)\rho^* \le \delta.$$

Thus we have from the above and (12.34) that

$$E\rho(X_0,Y_0) \le \rho(C,\mu)+3\delta.$$

This proves the existence of a finite window sliding-block encoder and a finite window length decoder with performance arbitrarily close to that achievable by block codes. □

The only use of ergodicity in the proof of the theorem was in the selection of the source sync sequence used to imbed the block code in a sliding-block code. The result would extend immediately to nonergodic stationary sources (and hence to nonergodic AMS sources) if we could somehow find a single source sync sequence that would work for all ergodic components in the ergodic decomposition of the source. Note that the source synch sequence affects only the encoder and is irrelevant to the decoder which looks for asynchronous codewords prefixed by channel synch sequences (which consisted of a single channel letter repeated several times). Unfortunately, one cannot guarantee the existence of a single source sequence with small but nonzero probability under all of the ergodic components. Since the components are ergodic, however, an infinite length sliding-block encoder could select such a source sequence in a simple (if impractical) way: proceed as in the proof of the theorem up to the use of Corollary 2.1. Instead of using this result, we construct by brute force a punctuation sequence for the ergodic component in effect. Suppose that $\mathcal{G} = \{G_i;\ i = 1, 2, \ldots\}$ is a countable generating field for the input sequence space. Given δ, the infinite length sliding-block encoder first finds the smallest value of i for which

$$0 < \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{G_i}(T^k x),$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{G_i}(T^k x)\rho(x_k, a^*) \le \delta,$$

that is, we find a set with strictly positive relative frequency (and hence strictly positive probability with respect to the ergodic component in effect) which occurs rarely enough to ensure that the sample average distortion between the symbols produced when G_i occurs and the reference letter is smaller than δ. Given N and δ there must exist an i for which these relations hold (apply the proof of Lemma 2.6 to the ergodic component in effect with γ chosen to satisfy (12.37) for that component and then replace the arbitrary set G by a set in the generating field having very close probability). Analogous to the proof of Lemma 2.6 we construct a punctuation sequence $\{Z_n\}$ using the event G_i in place of G. The proof then follows in a like manner except that now from the dominated convergence theorem we have that

$$E(1_2(Z_0)\rho(X_0,a^*)) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}E(1_2(Z_i)\rho(X_i,a^*)$$
$$= E(\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}1_2(Z_i)\rho(X_i,a^*)) \le \delta$$

by construction.

The above argument is patterned after that of Davisson and Gray [28] and extends the theorem to stationary nonergodic sources if infinite window sliding-block encoders are allowed. We can then approximate this encoder by a finite-window encoder, but we must make additional assumptions to ensure that the resulting encoder yields a good approximation in the sense of overall distortion. Suppose that f is the infinite window length encoder and g is the finite window-length (say $2L+1$) encoder. Let $\mathcal{G}$ denote a countable generating field of rectangles for the input sequence space. Then from Corollary 5.1 applied to $\mathcal{G}$ given $\epsilon > 0$ we can find for sufficiently large N a finite window sliding-block code $r : A^{2N+1} \to B$ such that $\Pr(r \ne f') \le \epsilon/(2L+1)$, that is, the two encoders produce the same channel symbol with high probability. The issue is when does this imply that $\rho(fg,\mu)$ and $\rho(rg,\mu)$ are therefore also close, which would complete the proof. Let $\overline{r} : A^{\mathcal{T}} \to B$ denote the infinite-window sliding block encoder induced by r, i.e., $\overline{r}(x) = r(x_{-N}^{2N+1})$. Then

$$\rho(fg,\mu) = E(\rho(X_0,\hat{X}_0)) = \sum_{b\in B^{2L+1}}\int_{x\in V_f(b)} d\mu(x)\rho(x_0,g(b)),$$

where

$$V_f(b) = \{x : f(x)^{2L+1} = b\}$$

and $f(x)^{2L+1}$ is shorthand for $f(x_i)$, $i = -L,\dots,L$, that is, the channel $(2L+1)$-tuple produced by the source using encoder x. We therefore have that

$$\rho(\overline{r}g,\mu) \le \sum_{b\in B^{2L+1}}\int_{x\in V_f(b)} d\mu(x)\rho(x_0,g(b))$$
$$+ \sum_{b\in B^{2L+1}}\int_{x\in V_{\overline{r}}(b)-V_f(b)} d\mu(x)\rho(x_0,g(b))$$
$$= \rho(f,\mu) + \sum_{b\in B^{2L+1}}\int_{x\in V_{\overline{r}}(b)-V_f(b)} d\mu(x)\rho(x_0,g(b))$$
$$\le \rho(f,\mu) + \sum_{b\in B^{2L+1}}\int_{x\in V_{\overline{r}}(b)\Delta V_f(b)} d\mu(x)\rho(x_0,g(b)).$$

By making N large enough, however, we can make

$$\mu(V_{\overline{r}}(f)\Delta V_f(b))$$

arbitrarily small simultaneously for all $b \in \hat{A}^{2L} + 1$ and hence force all of the integrals above to be arbitrarily small by the continuity of integration. With Lemma 12.6 and Theorem 12.6 this completes the proof of the following theorem.

Theorem 12.7. Theorem 11.7.2: *Given an AMS source μ and an additive fidelity criterion with a reference letter,*

$$\delta_{\mathrm{SBC}}(R, \mu) = \delta(R, \mu),$$

that is, the class of sliding-block codes is capable of exactly the same performance as the class of block codes.

The sliding-block source coding theorem immediately yields an alternative coding theorem for a code structure known as *trellis encoding* source codes wherein the sliding-block decoder is kept but the encoder is replaced by a tree or trellis search algorithm such as the Viterbi algorithm [44]. Details can be found in [53] and an example is discussed in Section 13.3.

12.9 A Geometric Interpretation

We close this chapter on source coding theorems with a geometric interpretation of the operational DRFs in terms of the $\overline{\rho}$ distortion between sources. Suppose that μ is a stationary and ergodic source and that $\{\rho_n\}$ is an additive fidelity criterion. Suppose that we have a nearly optimal sliding-block encoder and decoder for μ and a channel with rate R, that is, if the overall process is $\{X_n, \hat{X}_n\}$ and

$$E\rho(X_0, \hat{X}_0) \le \delta(R, \mu) + \delta.$$

If the overall hookup (source/encoder/channel/decoder) yields a distribution p on $\{X_n, \hat{X}_n\}$ and distribution η on the reproduction process $\{\hat{X}_n\}$, then clearly

$$\overline{\rho}(\mu, \eta) \le \delta(R, \mu) + \delta.$$

Furthermore, since the channel alphabet is B the channel process must have entropy rate less than $R = \log ||B||$ and hence the reproduction process must also have entropy rate less than B from Corollary 6.4. Since δ is arbitrary,

$$\delta(R, \mu) \ge \inf_{\eta: \overline{H}(\eta) \le R} \overline{\rho}(\mu, \eta).$$

Suppose next that p, μ and η are stationary and ergodic and that $\overline{H}(\eta) \le R$. Choose a stationary p having μ and η as coordinate processes such that

$$E_p \rho(X_0, Y_0) \le \overline{\rho}(\mu, \nu) + \delta.$$

We have easily that $\overline{I}(X;Y) \le \overline{H}(\eta) \le R$ and hence the left hand side is bounded below by the process distortion rate function $\overline{D}_s(R, \mu)$. From Theorem 9.1 and the block source coding theorem, however, this is just the operational distortion-rate function. We have therefore proved the following result [63].

Theorem 12.8. *Let μ be a stationary and ergodic source and let $\{\rho_n\}$ be an additive fidelity criterion with a reference letter. Then*

$$\delta(R, \mu) = \inf_{\eta: \overline{H}(\eta) \le R} \overline{\rho}(\mu, \eta), \tag{12.38}$$

that is, the operational DRF (and hence the distortion-rate function) of a stationary ergodic source is just the "distance" in the $\overline{\rho}$ sense to the nearest stationary and ergodic process with the specified reproduction alphabet and with entropy rate less than R.

Chapter 13
Properties of Good Source Codes

Abstract Necessary conditions for a source code to be optimal or a sequence of source codes to be asymptotically optimal for a stationary source are developed for block and sliding-block codes.

13.1 Optimal and Asymptotically Optimal Codes

In Eq. 5.14 the *operational distortion-rate function (DRF)* for the source μ, channel ν, and code classes $\mathcal{E}$ and $\mathcal{D}$ was defined by

$$\Delta(\mu, \nu, \mathcal{E}, \mathcal{D}) = \inf_{f \in \mathcal{E}, g \in \mathcal{D}} \Delta(\mu, f, \nu, g). \tag{13.1}$$

This chapter considers only source codes and hence the channel ν is assumed to be noiseless. A source code (f, g) is said to be *optimal* if it achieves the infimum, that is, if $f \in \mathcal{E}$, $g \in \mathcal{D}$, and

$$\Delta(\mu, f, \nu, g) = \Delta(\mu, \nu, \mathcal{E}, \mathcal{D}). \tag{13.2}$$

Optimal codes might not exist, but from the definition of infimum we can always get close. Hence we define a sequence (f_n, g_n), $n = 1, 2, \ldots$ to be *asymptotically optimal* or *a.o.* if

$$\lim_{n \to \infty} \Delta(\mu, f_n, \nu, g_n) = \Delta(\mu, \nu, \mathcal{E}, \mathcal{D}). \tag{13.3}$$

This chapter is concerned with developing the implications of a code being optimal or a sequence of codes being asymptotically optimal (which can be interpreted as looking at good or *nearly* optimal codes). Note that any property obtained for a.o. codes implies a result for optimal codes if they exist by setting $f_n = f$ and $g_n = g$ for all n. We usually consider optimal and asymptotically optimal separately since the former is simpler

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when it is applicable. These implications are in terms of necessary conditions for optimality or asymptotic optimality. The conditions describe attributes of encoders, decoders, and the distributions of the encoded and reproduction sequences. In the special case of squared-error distortion, the behavior of second order moments of the reproduction and the error sequence are quantified. We confine interest to stationary sources in order to keep things relatively simple.

In (12.1-12.2) the definition of operational rate-distortion function was specialized to the case of block codes of dimension N and code rate R and to block codes of arbitrary dimension and code rate R, definitions which we repeat here to have them handy:

$$\begin{aligned}\delta(R,\mu) &= \Delta(\mu,\nu,\mathcal{E},\mathcal{D}) = \inf_N \delta_N(R,\mu),\\ \delta_N(R,\mu) &= \inf_{C\in\mathcal{K}(N,R)} \rho(C,\mu),\end{aligned}$$

where ν is a noiseless channel as described in Section 12.2, $\mathcal{E}$ and $\mathcal{D}$ are classes of block codes for the channel, and $\mathcal{K}(N,R)$ is the class of all block length N codebooks C with

$$\frac{1}{N}\log||C|| \le R. \tag{13.4}$$

It was there argued that given a decoder of block codes of length N, an optimal encoder in the sense that no encoder could do better is given by a minimum distortion search of the decoder codebook. This observation is the original example (in Shannon [163]) of an optimality property of a source code — a necessary condition for a block code to be optimal is that the encoder be a minimum distortion (or "nearest neighbor") mapping, at least with probability 1. Shannon defined his source codes to have this property. Here we allow a more general definition of an encoder to show how a decoder can be optimized for a fixed (but not necessarily optimum) encoder, but observe a necessary condition for overall optimality is that the encoder have this property, that is, that the encoder be a minimum distortion search matched to the decoder. The introduction of this extra degree of freedom results in several useful properties, analogous to the introduction of the extra distribution η in the evaluation of rate-distortion functions which led to useful conditions for an optimization and an alternating optimization algorithm in Section 9.5. Block codes have other such optimality properties, many of which were first observed in the scalar (quantization) case by Lloyd [110] and in the vector case by Steinhaus [175].

13.2 Block Codes

Block source codes are also called block quantizers, multidimensional quantizers, and vector quantizers since they discretize or quantize a continuous (or large discrete) space into a relatively small discrete index set, and the indices are then decoded into approximations of the original input vector. While the emphasis of this book is on sliding-block codes, we treat the well-known optimality properties for block codes for two reasons. First, one way to prove sliding-block coding theorems is to embed a block code into a Rohlin tower to construct a stationarized version of the block code with approximately the same per-symbol average distortion. Thus having a good block code can lead to a good sliding-block code. Some proofs of the sliding-block coding theorems avoid directly using block codes by taking advantage of results from ergodic theory, but the ergodic theory results usually use block constructions in their proofs as well. Second, the optimality properties of block codes are simpler to state and prove and they provide some interesting comparisons with the stationary code results of the next section.

This section treats well-known properties and methods for developing the properties for vector quantizers with notational changes as needed to be consistent with the book. The reader is referred to [50, 71] for further discussion. The usual formulation for a block source coder for a source X with alphabet A_X involves two mappings, an encoder $\alpha : A_X^N \to \mathbb{I}$, where $\mathbb{I}$ is an index set of size $M = \|\mathbb{I}\|$, and a decoder $\beta : \mathbb{I} \to C$, where it is usually assumed that $\mathbb{I}$ is either a set of integers $\{0, 1, \ldots, M-1\}$ (common when the block code is considered outside the context of a communications channel) or is a sequence space of channel symbols $\mathbb{I} = A_U^K$, where $A_U = \{0, 1\}$ for binary channel codes or A_U is some other finite set of the form $\{0, 1, \ldots, m-1\}$, in which case $M = \|A_U\|^K$.

The collection of reproduction words C is called the *reproduction codebook* (or simply *codebook* if the usage is clear from context) and it is usually assumed for convenience that the decoder is a one-to-one mapping of indices into distinct reproduction codewords so that

$$C = \{\beta(i); i \in C\}$$

so that $\|C\| = M$. The *code rate* or *transmission rate* of the code is defined by

$$R = \log M,$$

where the units are bits per input vector if the logarithm is base 2 and nats per vector if it is base e. It is common to consider the normalized or per-symbol code rate of

$$R = \frac{1}{N} \log M.$$

Context should make clear if it is the normalized or unnormalized rate being considered. In this section we emphasize a fixed N and hence usually do not normalize. If $\mathbb{I} = A_U^K$, then the normalized rate is

$$R = \frac{K}{N} \log \|A_U\|.$$

In the important simple special case where the code is binary and $\|A_U\| = 2$, then $R = K/N$, the number of binary channel symbols produced by the encoder for each N channel symbols put into it.

We simplify the vector notation in this section by dropping the subscripts N and assume that X is a random vector with sample values x chosen in a vector alphabet A_X. Rates will not be normalized and the dimension will be implicit.

An encoder α is equivalent to a partition of the vector input space A_X defined by $\mathcal{P} = \{P_i; i \in \mathbb{I}\}$, where

$$P_i = \{x \in A_X : \alpha(x) = i\}.$$

The partition notation provides a useful representation of the encoder as

$$\alpha(x) = \sum_{i \in \mathbb{I}} 1_{P_i}(x).$$

Similarly a decoder β is described by its reproduction codebook C.

Assuming that no errors are made in the transmission of the channel codeword, then for most properties the specific nature of the index set $\mathbb{I}$ is unimportant and all that matters is the number of elements M and the codewords in C.

The combined operation of an encoder and decoder is often referred to simply as a *quantizer* or *vector quantizer*. We will use Q to denote both the pair $Q = (\alpha, \beta)$ and the overall operation defined by

$$Q(x) = \beta(\alpha(x)).$$

Two quantizers will be said to be equivalent if they have the same index set $\mathbb{I}$ and yield the same overall mapping Q. The rate of the quantizer (unnormalized) is given by $R = R(Q) = \log(\alpha(A_X))$, the log of the size of the index set (or reproduction codebook).

The principal goal in the design of vector quantizers is to find a codebook (decoder) and a partition (encoder) that minimizes an average distortion with respect to a distortion measure d. The average distortion for a vector quantizer $Q = (\alpha, \beta)$ applied to a random vector X with distribution μ is

$$\Delta(\alpha, \beta) = E[\rho(X, Q(X))] = \int \rho(x, Q(x)) d\mu(x). \tag{13.5}$$

Define as earlier the operational distortion-rate function $\Delta(\mu)$ by

$$\Delta(R) = \inf_{\alpha,\beta:R(\alpha)\le R} \Delta(\alpha,\beta). \tag{13.6}$$

A code is *optimal* if

$$\Delta(\alpha,\beta) = \Delta(R).$$

Recall that at present everything is for a fixed dimension N and that the clutter of more notation will be necessary to make N explicit when it is allowed to vary, for example if we wish to quantify the long term performance when the block code is applied to an AMS source as in Lemma 12.1.

The most fundamental of the conditions for optimality of a quantizer (α,β) follows from the obvious inequality

$$\begin{aligned}\Delta(\alpha,\beta) &= E[\rho(X,Q(X))] = \int \rho(x,Q(x))d\mu(x)\\ &\ge \int \min_{y\in C} \rho(x,y)d\mu(x),\end{aligned} \tag{13.7}$$

where the minimum exists since the codebook is assumed finite. This unbeatable lower bound to the average distortion for a quantizer with reproduction codebook C and hence for a given decoder is achieved with equality if the encoder is defined to be the minimum distortion encoder:

$$\alpha(x) = \operatorname*{argmin}_{i\in\mathbb{I}} \rho(x,\hat{x}_i), \tag{13.8}$$

where $C = \{\hat{x}_i; i \in \mathbb{I}\}$. The encoder is not yet well defined in a strict sense because there can be ties in distortion, in which case the encoder has to chose among multiple indices yielding the same distortion. In this case any tie-breaking rule can be used without affecting the distortion, for example choose the index lowest in lexicographical order. It can be assumed that the optimal encoder is of this form since, if it were not, changing to a minimum distortion encoder can not increase the average distortion. In the classic paper on source coding with a fidelity criterion [163], Shannon assumed that encoders were of this form. We do not make that assumption since it is useful to consider performance of a quantizer as a function of decoder (codebook) and encoder (partition) separately, but when all is said and done the best choice for an encoder (in terms of minimizing average distortion) is the minimum distortion encoder.

An alternative means of describing an optimal encoder is in terms of the encoder partition by

$$P_i \subset \{x : \rho(x,\hat{x}_i) \le \rho(x,\hat{x}_j); j \ne i\},$$

where again some tie-breaking rule is needed to assign an index to points on the border.

Next suppose that the encoder α or its partition $\mathcal{P}$ is specified. Given any subset $S \subset A_X$ of probability $\mu(S) > 0$, the point $z \in \hat{A}$ (if it exists) for which

$$E[d(X, z) \mid X \in S] = \inf_{y \in \hat{A}} E[d(X, y) \mid X \in S]$$

is called the *centroid* of S and denoted by $\text{cent}(S)$. If the set S has zero probability, then the centroid can be defined in an arbitrary fashion. The name reflects the origins of the word as the centroid or center of gravity of a set in Euclidean space with respect to the squared-error distortion and Lebesgue measure. If the centroid exists, then

$$\text{cent}(S) = \operatorname*{argmin}_{y \in \hat{A}} E[d(X, y) \mid X \in S].$$

Centroids exist for many distortion measures and sets of interest. For example, if the $A_X = \mathbb{R}^N$ and the distortion is additive squared error (the square of the ℓ_2 norm of the vector difference), then the centroid is given by the conditional expectation $E[X \mid X \in S]$, the minimum mean-squared estimate of the source vector given the event $X \in S$. Other interesting distortion measures with centroids are considered in [50]. If the appropriate centroids exist, then the properties of conditional expectation yield the inequality

$$\begin{aligned} E[d(X, Q(X))] &= \sum_{i \in \mathbb{I}} E[d(X, \hat{x}_i) 1_{P_i}(X)] \\ &= \sum_{i \in \mathbb{I}} E[d(X, \hat{x}_i) \mid X \in P_i] \mu(P_i) \\ &\geq \sum_{i \in \mathbb{I}} \text{cent}(P_i) \mu(P_i), \end{aligned} \tag{13.9}$$

which holds with equality if the decoder output or reproduction codeword assigned to i is the centroid of P_i. In the squared error case this has the intuitive interpretation of being the minimum mean-squared estimate of the input given the received index.

These two conditions were developed for squared error for vectors by Steinhaus [175] and in the scalar case for more general distortion measures by Lloyd [110] and they have since become known as the Lloyd conditions and quantizers which satisfy the conditions are sometimes referred to as Lloyd-optimal quantizers, although satisfaction of the two conditions does not ensure global optimality. Each condition, however, ensures conditional optimality given the other component of the quantizer and hence, as both Lloyd and Steinhaus observed, they provide an iterative algorithm for improving the quantizer performance. Begin-

ning with a distribution, distortion measure, and initial reconstruction codebook, the optimal encoder is a minimum distortion mapping. The decoder can then be replaced by the centroids. Each step can only decrease or leave unchanged the average distortion, hence the algorithm is a descent algorithm with respect to average distortion. The distribution can be an empirical distribution based on a training set of data, which makes the algorithm an early example of clustering and statistical (or machine) learning. The idea has been rediscovered in many fields, perhaps most famously a decade later as the "k-means" clustering algorithm of MacQueen [112]. This was one of the first examples of what has since become known as an alternating optimization (AO) algorithm, where a complicated optimization (e.g., nonconvex) can be broken down into two separate simpler optimizations [14]. Taken together, the conditions yield the following lemma.

Lemma 13.1. Lloyd Quantizer Optimality Properties *An optimal quantizer must satisfy the following two conditions (or be equivalent to a quantizer which does):*

Optimum encoder for a given decoder *Given a decoder with reproduction codebook $C = \{\hat{x}_i; i \in \mathbb{I}\}$, the optimal encoder satisfies*

$$\alpha(x) = \operatorname*{argmin}_{i \in \mathbb{I}} \rho(x, \hat{x}_i). \tag{13.10}$$

Optimum decoder for a given encoder *Given an encoder α, the optimal decoder satisfies*

$$\beta(i) = \operatorname{cent}(\{x : \alpha(x) = i\}). \tag{13.11}$$

The implications of these optimality properties for the design of vector quantizers are explored in depth in the literature. For example, see [50, 61, 115, 128, 152, 71]. Two observations merit making. The decoder for a block code is simple in principal, it is simply a table lookup. A channel codeword provides an index and the output is the reproduction codeword in the codebook with that index. The encoder, however, requires a minimum distortion search of the reproduction codebook to find the best fit to the observed input vector. Search can be costly, however, especially as the dimension grows. There is a large literature on techniques for avoiding a full search of a codebook, from forcing a structure on the codebook (such as a tree or trellis or as a linear combination of basis vectors) to performing only a partial or approximate search. See, e.g., [71].

Moment Properties

In the special case of the squared-error distortion where

$$d(x, y) = \|x - y\|^2 = \sum_{i=0}^{N-1} (x_i - y_i)^2,$$

where N is the dimension of the vectors x, y, optimal quantizers have several additional interesting properties in terms of their moments. In fact, the quantizers need not be optimal to have these properties, the only requirement is that they satisfy the centroid condition so that the codewords are given by

$$\hat{x}_i = E[X \mid \alpha(X) = i], i \in \mathbb{I}.$$

The following lemma collects these conditions. The results are simple consequences of basic properties of vector linear prediction. The lemma follows [50] Lemma 11.2.2. See also the scalar special case in Lemma 6.2.2.

Lemma 13.2. *A vector quantizer which satisfies the centroid condition for the squared-error distortion measure has the following properties:*

1. $E(Q(X)) = E(X)$
2. $E(X^t Q(X)) = E(\|Q(X)\|^2)$
3. $E((X - Q(X))^t Q(X)) = 0$
4. $E(\|Q(X)\|^2) = E(\|X\|^2) - E(\|X - Q(X)\|^2)$.

Proof. Let $\mathcal{P} = \{P_i, i \in \mathbb{I}\}$ denote the partition associated with the partition and $C = \{\hat{x}_i; i \in \mathbb{I}\}$ the reproduction codebook. By assumption, the $\hat{x}_i$ are the centroids of the the partition cells P_i, which for squared error are the conditional expectations of X given $X \in P_i$. Using conditional expectation,

$$\begin{aligned} E(X) &= \sum_E \left(X \sum_{i \in \mathbb{I}} 1_{P_i}(X) \right) = \sum_{i \in \mathbb{I}} E(X 1_{P_i}(X)) \\ &= \sum_{i \in \mathbb{I}} \mu_X(P_i) E(X \mid X \in P_i) = \sum_{i \in \mathbb{I}} \mu_X(P_i) \hat{x}_i = E[Q(X))], \end{aligned}$$

proving the first property. Since

$$Q(X) = \sum_{i \in \mathbb{I}} \hat{x}_i 1_{P_i}(X),$$

we have that

$$
\begin{aligned}
E(X^t Q(X)) &= E(X^t \left(\sum_{i \in \mathbb{I}} \hat{x}_i 1_{P_i}(X) \right)) = \sum_{i \in \mathbb{I}} \left(E(1_{P_i}(X) X^t) \hat{x}_i \right) \\
&= \sum_{i \in \mathbb{I}} \left(\mu_X(P_i) E(X^t \mid X \in P_i) \hat{x}_i \right) = \sum_{i \in \mathbb{I}} \left(\mu_X(P_i) \hat{x}_i^t \hat{x}_i \right) \\
&= \sum_{i \in \mathbb{I}} \mu_X(P_i) \hat{x}_i^t \hat{x}_i = \|Q(X)\|^2,
\end{aligned}
$$

proving the second property. The third property follows from the second. The final property follows from expanding the left hand side and using the second and third properties

$$
\begin{aligned}
E(\|X - Q(X)\|^2) &= E\left((X - Q(X))^t (X - Q(X)) \right) \\
&= E\left(X^t (X - Q(X)) \right) - E\left(Q(X)^t (X - Q(X)) \right) \\
&= E(\|X\|^2) - E(X^t Q(X)) - 0 \\
&= E(\|X\|^2) - E(\|Q(X)\|^2).
\end{aligned}
$$

. □

The lemma has the intuition that the centroid condition is sufficient to ensure that the quantizer is an unbiased estimator of the input given the index (or the quantized value itself). The second property shows that the correlation of the quantizer output and the input equals the energy in the quantizer output. In particular, the input and output are *not uncorrelated*. and hence can not be independent. This conflicts directly with the frequently assumed model in the communications and signal processing literature where quantizer error is treated as signal-independent white noise. The third property shows that the error and the estimate quantizer output (which is an estimate of the input given the quantizer index) have 0 correlation, which is simply an example of the orthogonality property since, for a fixed encoder, $Q(X)$ is an optimal linear estimate for X given $Q(X)$.

13.3 Sliding-Block Codes

A sliding-block code (f, g) for source coding is said to be optimum if it yields an average distortion equal to the operational distortion-rate function, $\Delta(f, g) = \Delta_X(R)$. Unlike the simple scalar quantizer case (or the nonstationary vector quantizer case), however, there are no simple conditions for guaranteeing the existence of an optimal code. Hence usually it is of greater interest to consider codes that are asymptotically optimal in the sense that their performance approaches the optimal in the limit, but there might not be a code which actually achieves the limit. Before

considering asymptotic optimality, consider the natural extensions of the quantizer optimality properties to a sliding-block code. Sliding-block decoders do have a basic similarity to quantization decoders in that one can view the contents of the decoder shift register as an index i in an index set $\mathbb{I}$ and the decoder is a mapping of $\mathbb{I}$ into a single reproduction symbol $g(i) \in \hat{A}$, say. Usually in practical systems the decoder has finite length, in which case the index set $\mathbb{I}$ is finite. If the encoder is given, then the mapping of inputs into indices at a particular time is fixed, and hence the centroid condition must still hold if the centroids exist for the distortion measure being used. This yields the following lemma.

Lemma 13.3. *If the encoder f in a sliding-block source coder is fixed and results in decoder shift register contents $U \in \mathbb{I}$ at time 0, then a necessary condition for the decoder g to be optimal with respect to the encoder is that*

$$g(i) = \text{cent}(X_0 \mid U = i) \equiv \operatorname*{argmin}_{y \in \hat{A}} E\left(\rho(X_0, y) \mid U = i\right) \tag{13.12}$$

The lemma follows in the same manner as the quantizer result, and as in that case it implies for the squared-error distortion case that $g(i)$ be the conditional expectation of the input at the same time given the contents of the shift register and knowledge of the encoder mapping of inputs into indexes. The application of Lloyd's centroid condition for quantizers to sliding-block decoders was first treated by Stewart [176, 177]. This property can be used to tune a decoder to a training sequence using an empirical conditional expectation.

Unfortunately the corresponding result for the encoder — that the minimum distortion rule is the optimal encoder for a fixed decoder — does not have a simple extension to sliding-block codes. There is, however, a hybrid system that couples a sliding-block decoder with a block encoder which does resemble the Lloyd alternating optimization for vector quantization, and it has the added advantage that the minimum distortion search required of the encoder does not require computational complexity increasing exponentially with the block length of the code. The trick is that the sliding-block coder structure allows a low complexity minimum distortion search. The technique is known as *trellis source encoding* and it was introduced by by Viterbi and Omura [188] and the connections with sliding-block codes and the Lloyd iteration were developed by Stewart et al. [53, 176, 177]. A brief overview is presented here to illustrate the similarities among the Lloyd conditions, sliding-block, and block codes. Details can be found in the cited references.

To keep things simple, we focus on one bit per symbol codes so that the noiseless channel alphabet is binary and one reproduction symbol is produced for each channel bit. Consider the simple sliding-block code of Figure 2.1 with a more general decoder function g as in Fig-

ure 13.1. The temporal operation of the decoder can be depicted as in

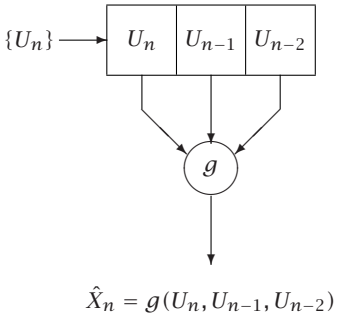

Fig. 13.1 A simple sliding-block decoder

Figure 13.2, a directed graph called a *trellis*. The decoder is viewed as a finite-state machine where at time n the state consists of received channel symbols in the shift register except for the most recent — in this case $S_n = (U_{n-1}, U_{n-2})$ so that there are four possible states $\{00, 01, 10, 11\}$. The states at a particular time are represented by the darkened circles stacked vertically and labeled on the far left. If at time n the shift register is in state $s_n = (u_{n-1}, u_{n-2})$ and a channel symbol u_n is received, then the output will be $g(u_n, u_{n-1}, u_{n-2})$ and the state will change to $s_{n+1} = (u_n, u_{n-1})$. Thus the next-state rule given the current state and the current received channel symbol can be described in a state transition table as in Table 13.1. In general there is a next state rule

s_n	11	10	01	00	11	10	00	01
u_n	1	1	1	1	0	0	0	0
s_{n+1}	11	11	10	10	01	01	00	00

Table 13.1 State transition table

$s_{n+1} = r(u_n, s_n)$. The state transistions are noted in the trellis by connecting the states between times by a *branch* which is labeled by the decoder output produced by the transition between the two states connected by the branch. In the figure the upper branch shows the transition if the channel symbol is a 1, the lower branch shows the transition if the channel symbol is a 2. The picture for time n is replicated at every time instant. The leftmost column of states can be considered as time 0 and the trellis continues to replicate to the right. Continuing with this simple example, suppose that the decoder g is fixed, and we want to design a good encoder. While the end goal may be another sliding-block code to match the theoretical emphasis, suppose for the time being that a block

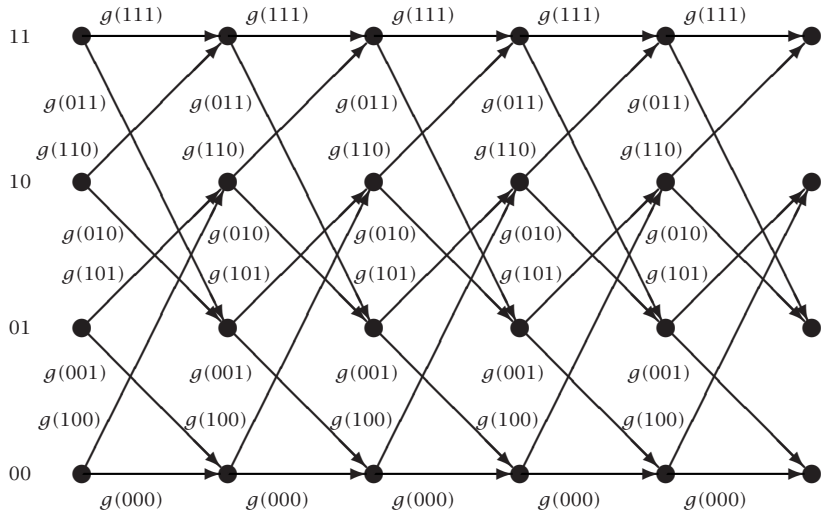

Fig. 13.2 Trellis diagram of sliding-block code

encoder is to be used. In that case, the trellis provides a picture of all of the reproduction sequences available to the decoder, so the general principal would be to observe an input sequence of length, say, L, and find the best possible sequence of trellis branch labels through the trellis for L steps, that is, find the minimum distortion path through the trellis. An immediate issue how to initiate the encoding algorithm. In particular, if we allow the search algorithm to consider all possible initial states, then at the beginning of the block the decoder must be told what state to begin in, that is, what paths through the trellis are allowed. The optimal choice might be to place no constraint on the start, but this means that many bits would need to be sent at the beginning of a block, while only one bit per input sample would be required thereafter. The usual practical solution is to pick an arbitrary initial state, say the all-zero state, to begin with. Then the path through the trellis can be sent with one bit per branch for the current and future blocks. Hopefully the effects of an arbitrary and possibly bad initial state will wash out with time.

Given that the encoder and decoder initialize their states to a common state, say σ_0, then as in the block source code case, the optimal encoder for a source block x^L of length L will choose the sequence of bits u^L that drives the decoder through the trellis so as to yield the smallest possible total distortion; that is, the encoder will find

$$\operatorname*{argmin}_{u^L} \sum_{i=0}^{L-1} \rho(x_i, g(u_i, s_i)).$$

where the s_i are computed from the previous states and the inputs using Table 13.1. What makes this structure so useful is that instead of having to compute the distortions for all 2^L possible reproduction L tuples with respect to an observed input tuple, the minimum distortion path can be found sequentially in time by a simple algorithm, as will be seen. An immediate point is that if the decoder is a good sliding-block code, then a natural choice for an encoder is a minimum distortion search through all possible decoder sequences, that is, find the binary channel sequence that will drive the decoder through the sequence of outputs that provide a good match to an observed input sequence.

The 2^L possible reproduction L-tuples correspond to the 2^L possible sequences of reproduction symbols or branch labels resulting from the 2^L possible binary path maps through the trellis from the initial state to one of the possible final states. A brute force minimization would be to compute for each of these 2^L path maps and their distortion with respect to the input x^L and choose the binary sequence which results in the minimum distortion sequence of branch labels. The trellis structure decreases the work, however, by eliminating the need for computing the distortion for *all* possible paths through the trellis. Many of the paths are bad and can not be candidates for the best path. Suppose that we know the best paths into each of the 2^{K-1} possible states at time $n = L - 1$ along with the resulting total distortion resulting from the associated sequence of reproductions/branch labels. Suppose that

$$\upsilon^L \equiv \operatorname*{argmin}_{u^L} \sum_{i=0}^{L-1} \rho(x_i, g(u_i, s_i))$$

is the best binary path map and that the resulting distortion is

$$\Delta_L = \sum_{i=0}^{L-1} \rho(x_i, g((\upsilon^L)_i, s_i)) = \min_{u^L} \sum_{i=0}^{L-1} \rho(x_i, g(u_i, s_i)),$$

where as always the state sequence is determined from the path map. As the notation is cluttered enough already, this dependence is not shown explicitly. Consider a time $n < L$. An optimal overall path map υ^L must have resulted in the decoder being in some particular state, say s, at time n, and in a binary path map $\upsilon^n(s)$ that resulted in the decoder being in the state s at time n, and in a running distortion of

$$\Delta_n(s) = \sum_{i=0}^{n-1} \rho(x_i, g((\upsilon^L)_i, s_i)),$$

with $s_n = s = r(u_{n-1}, s_{n-1})$. Furthermore, this path *must have been the best path from the initial state σ_0 through the state s at time n*; that is,

$$(v_0^L, v_1^L, \ldots, (v^L)_{n-1}) = v^n(s) \equiv \operatorname*{argmin}_{u^n} \sum_{i=0}^{n-1} \rho(x_i, g(u_i, s_i))$$

$$\Delta_n = \sum_{i=0}^{n-1} \rho(x_i, g((v^L)_i, s_i)) = \min_{u^n} \sum_{i=0}^{n-1} \rho(x_i, g(u_i, s_i)).$$

This follows from the additive nature of distortion since if the length n prefix of v^L, $((v^L)_0, (v^L)_1, \ldots, (v^L)_{n-1})$, were not the minimum distortion path from the intitial state to state s at time n, then there would be another path into this node with strictly smaller distortion. Were that the case, however, that other path would be a better prefix to the overall path and it would yield smaller total distortion by yielding smaller distortion in the first n-tuple and no worse distortion for the remainder of the path. The remainder of the path is not changed because the choices from time n on depend only on the state of the decoder at time n. This argument assumes that we know the state through which the optimal path passes. In general we can only say before the optimal path is known that at time n the decoder must be in *some* state. This implies, however, that at time n at each state all we need track is what the best path so far into that state is and what the corresponding distortion is. All inferior paths into the state can be discarded as no longer being candidates for prefix of an optimal path map. This yields a search algorithm for finding the minimum distortion path through the trellis, which can be described informally as follows.

Step 0 Given: input sequence x^N, length K sliding-block decoder g, initial state σ_0. State space S = all 2^{K-1} binary $(K-1)$-tuples. Next state mapping: If the old state is $s = (b_0 b_1 \cdots, b_{k-2}, b_{K-1})$ and the received channel symbol is u, then the next state is $r(u, s) = (u, b_0, b_1, \cdots, b_{k-2})$. Define $v^0(s)$ to be the empty set for all $s \in S$. Define $\Delta_n(s) = 0$ for all $s \in S$. Set $n = 1$.

Step 1 For each $s \in S$:

There can be at most two previous states $s_{n-1} = \sigma_0$ and σ_1, say, for which $s_n = s = r(0, \sigma_0) = r(1, \sigma_1)$ and for which s_{n-1} is reachable from the initial state. (For $n \geq K-1$ all states at time n are reachable from the initial state.) If there are two such states, compare $\delta_0 = \Delta_{n-1}(\sigma_0) + \rho(x_n, g(0, \sigma_0))$ with $\delta_1 = \Delta_{n-1}(\sigma_1) + \rho(x_n, g(1, \sigma_1))$. If $\delta_0 \leq \delta_1$, then set

$$v^n(s) = (v^{n-1}(\sigma_0), 0)$$
$$\Delta_n(s) = \Delta_{n-1}(\sigma_0) + \rho(x_n, g(0, \sigma_0)),$$

otherwise set

$$\upsilon^n(s) = (\upsilon^{n-1}(\sigma_1), 1)$$
$$\Delta_n(s) = \Delta_{n-1}(\sigma_1) + \rho(x_n, g(1, \sigma_1)).$$

That is, choose the minimum distortion path available from the choice of two paths entering state s at time n. This extends one of the two candidate paths of length $n-1$ available at the two allowed previous states and extinguishes the other, which can not be the suffix of an optimal path.
If there is only one allowed previous state σ, choose the path from that single state using the update formula above. If there is no allowed previous state, do nothing (there is no update to a best path or associated distortion to the given state at time n).
Set $n \leftarrow n+1$.

Step 2 If $n < L$, go to Step 1. If $n = L$, we have the best path maps $\upsilon^L(s)$ and corresponding distortions $\Delta_L(s)$ at time $L-1$ for all states $s \in S$. Set

$$s^* = \operatorname*{argmin}_{s \in S} \Delta_L(s)$$

and finish with binary path map $\upsilon^L(s^*)$ as the encoded bit sequence to be communicated to the receiver to drive the decoder to produce the reproduction.

Instead of computing 2^L separate distortions for L-tuples, the algorithm computes 2^{K-1} incremental distortions at each time (level in the trellis) and does the corresponding addition to maintain and store the cumulative distortion up to that time. This is done for each of the L levels of the trellis. Thus the algorithm complexity grows roughly linearly and not exponentially in the encoder block size L, but it does grow with the state space size and hence exponentially with the decoder shift register length. Thus a relatively small decoder shift register length with a large block length can yield an overall reasonable complexity. As of this writing (2010) shift register lengths of over 20 and block lengths of a million are reasonable. Decoding is of minimal complexity.

The basic optimality principle used here as that of dynamic programming, and its applications to channel coding and source coding were introduced by Andrew Viterbi. The algorithm is widely known as the Viterbi algorithm in communications and signal processing. See, e.g., [44].

At this point we have a hybrid code with a sliding block decoder and a block encoder. The block encoder can be made stationary using occasional input-dependent spacing as in the theoretical constructions of sliding-block codes from block codes, but in practice the Viterbi algorithm is usually run on very long blocks. There are many variations on the basic approach.

Asymptotically Optimal Sliding-Block Codes

This subsection largely follows Mao, Gray, and Linder [117, 72]. A sequence of rate-R sliding-block codes f_n, g_n, $n = 1, 2, \ldots$, for source coding is *asymptotically optimal* (a.o.) if

$$\lim_{n\to\infty} \Delta(f_n, g_n) = \Delta_X(R) = D_X(R). \tag{13.13}$$

An optimal code (when it exists) is trivially asymptotically optimal and hence any necessary condition for an asymptotically optimal sequence of codes also applies to a fixed code that is optimal by simply equating every code in the sequence to the fixed code.

Similarly, a simulation code g is optimal if $\overline{\rho}(\mu_X, \mu_{\overline{g}(Z)}) = \Delta(X|Z)$ and a sequence of codes g_n is asymptotically optimal if

$$\lim_{n\to\infty} \overline{\rho}(\mu_X, \mu_{\overline{g}_n(Z)}) = \Delta_{X|Z}. \tag{13.14}$$

Process approximation

The following lemma provides necessary conditions for asymptotically optimal codes. The results are a slight generalization and elaboration of Theorem 1 of Gray and Linder [72] as given in [117].

Lemma 13.4. *Given a real-valued stationary ergodic process X, suppose that f_n, g_n $n = 1, 2, \ldots$ is an asymptotically optimal sequence of stationary source codes for X with encoder output/decoder input alphabet B of size $\|B\| = 2^R$ for integer rate R. Denote the resulting reproduction processes by $\hat{X}^{(n)}$ and the B-ary encoder output/decoder input processes by $U^{(n)}$. Then*

$$\begin{aligned}
\lim_{n\to\infty} \overline{\rho}(\mu_X, \mu_{\hat{X}^{(n)}}) &= D_X(R) \\
\lim_{n\to\infty} H(\hat{X}^{(n)}) &= \lim_{n\to\infty} H(U^{(n)}) = R \\
\lim_{n\to\infty} \overline{d}(U^{(n)}, Z) &= 0,
\end{aligned}$$

where Z is an IID equiprobable process with alphabet size 2^R.

These properties are quite intuitive:

- The process distance between a source and an approximately optimal reproduction of entropy rate less than R is close to the Shannon distortion rate function. Thus frequency-typical sequences of the reproduction should be as close as possible to frequency-typical source sequences.

- The entropy rate of an approximately optimal reproduction and of the resulting encoded B-ary process must be near the maximum possible value in order to take advantage of all possible available information.
- The sequence of encoder output processes approaches an IID equiprobable source in the Ornstein process distance. If $R = 1$, the encoder output bits should look like fair coin flips.

Proof. The encoded and decoded processes are both stationary and ergodic since the original source is. From (12.38) and the source coding theorem,

$$\begin{aligned}\Delta(f_n, g_n) &= E[d(X_0, X_0^{(n)})] \geq \overline{\rho}(\mu_X, \mu_{\hat{X}^{(n)}}) \\ &\geq \inf_{\nu: H(\nu) \leq R} \overline{\rho}(\mu_X, \nu) = \Delta(\mu, R) = D_X(R).\end{aligned}$$

The first inequality follows since the $\overline{\rho}$ distance is the minimum average distortion over all couplings yielding the marginal distributions for X_0 and $X_0^{(n)}$. The second inequality follows since stationary coding reduces entropy rate so that $R \geq H(U^{(n)}) \geq H(\hat{X}^{(n)})$. Since the leftmost term converges to the rightmost, the first equality of the lemma is proved. From Lemma 8.4

$$R \geq H(\hat{X}^{(n)}) \geq I(X, \hat{X}^{(n)}).$$

From the process definition of the rate-distortion function, the dual to the process definition of the distortion-rate function (the rate-distortion formulation can be found in [119, 63, 76]), $I(X, \hat{X}^{(n)}) \geq R_X(\Delta(f_n, g_n))$. Taking the limit as $n \to \infty$, $R_X(\Delta(f_n, g_n))$ converges to R since the code sequence is asymptotically optimal and the Shannon rate-distortion function is a continuous function of its argument except possibly at $D = 0$, the dual of Lemma 9.1. Thus $\lim_{n\to\infty} H(U^{(n)}) = \lim_{n\to\infty} H(\hat{X}^{(n)}) = R$. proving the second equality of the lemma.

From Marton's inequality of Corollary 6.6,

$$N^{-1}\overline{d}_N(\mu_{U^N}, \mu_{Z^N}) \leq \left[\frac{\ln 2}{2N}(NR - H(U^N))\right]^{1/2}$$

and taking the limit as $N \to \infty$ using property (a) of Theorem 5.2 yields

$$\overline{d}(\mu_U, \mu_Z) \leq \left[\frac{\ln 2}{2}(R - H(U))\right]^{1/2}.$$

Applying this to $U^{(n)}$ and taking the limit using the previous part of the lemma completes the proof. □

If X is a B-process, then a sequence of a.o. simulation codes g_n yielding a reproduction processes $\tilde{X}^{(n)}$ satisfies $\lim_{n\to\infty} \overline{\rho}(\mu_X, \mu_{\tilde{X}^{(n)}}) = \Delta_{X|Z} = D_X(R)$ and a similar argument to the proof of the previous lemma implies that $\lim_{n\to\infty} H(\hat{X}^{(n)}) = H(Z) = R$.

Moment conditions

The next set of necessary conditions concerns the squared-error distortion and resembles the standard result for scalar and vector quantizers described in Lemma 13.2. The proof differs, however, in that in the quantization case the centroid property is used, while here simple ideas from linear prediction theory accomplish a similar goal. Define in the usual way the covariance $\mathrm{COV}(X,Y) = E[(X - E(X))(Y - E(Y))]$.

Lemma 13.5. *Given a real-valued stationary ergodic process X, suppose that f_n, g_n is an asymptotically optimal sequence of codes (with respect to squared error) yielding reproduction processes $\hat{X}^{(n)}$ with entropy rate $H(\hat{X}^{(n)}) \le R$, then*

$$\lim_{n\to\infty} E(\hat{X}_0^{(n)}) = E(X_0) \tag{13.15}$$

$$\lim_{n\to\infty} \frac{COV(X_0, \hat{X}_0^{(n)})}{\sigma^2_{\hat{X}_0^{(n)}}} = 1 \tag{13.16}$$

$$\lim_{n\to\infty} \sigma^2_{\hat{X}_0^{(n)}} = \sigma^2_{X_0} - D_X(R). \tag{13.17}$$

Defining the error as $\epsilon_0^{(n)} = \hat{X}_0^{(n)} - X_0$, then the necessary conditions become

$$\lim_{n\to\infty} E(\epsilon_0^{(n)}) = 0 \tag{13.18}$$

$$\lim_{n\to\infty} E(\epsilon_0^{(n)} \hat{X}_0^{(n)})) = 0 \tag{13.19}$$

$$\lim_{n\to\infty} \sigma^2_{\epsilon_0^{(n)}} = D_X(R). \tag{13.20}$$

The results are stated for time $k = 0$, but stationarity ensures that they hold for all times k.

Proof: For any encoder/decoder pair (f_n, g_n) yielding a reproduction process $\hat{X}^{(n)}$

$$\begin{aligned}\Delta(f_n, g_n) &\ge \inf_{a,b\in\mathbb{R}} \Delta(f_n, a g_n + b) \\ &\ge D_X(R) = \inf_{f,g} \Delta(f, g)\end{aligned}$$

where the second inequality follows since scaling a sliding-block decoder by a real constant and adding a real constant results in another sliding-block decoder with entropy rate no greater than that of the input. The minimization over a and b for each n is solved by standard linear prediction techniques as

$$a_n = \frac{\mathrm{COV}(X_0, \hat{X}_0^{(n)})}{\sigma^2_{\hat{X}_0^{(n)}}} \tag{13.21}$$

$$b_n = E(X_0) - a_n E(\hat{X}_0^{(n)}), \tag{13.22}$$

$$\begin{aligned} \inf_{a,b} \Delta(f_n, ag_n + b) &= \Delta(f_n, a_n g_n + b_n) \\ &= \sigma^2_{X_0} - a_n^2 \sigma^2_{\hat{X}_0^{(n)}}. \end{aligned} \tag{13.23}$$

Combining the above facts we have that since (f_n, g_n) is an asymptotically optimal sequence,

$$\begin{aligned} D_X(R) &= \lim_{n\to\infty} \Delta(f_n, g_n) \geq \lim_{n\to\infty} \Delta(f_n, a_n g_n + b_n) \\ &\geq D_X(R) \end{aligned} \tag{13.24}$$

and hence that both inequalities are actually equalities. The final inequality (13.24) being an equality yields

$$\lim_{n\to\infty} a_n^2 \sigma^2_{\hat{X}_0^{(n)}} = \sigma^2_{X_0} - D_X(R). \tag{13.25}$$

Application of asymptotic optimality and (13.21) to

$$\begin{aligned} \Delta(f_n, g_n) &= E\left((X_0 - \hat{X}_0^{(n)})^2\right) \\ &= E\left(([X_0 - E(X_0)] - [\hat{X}_0^{(n)} - E(\hat{X}_0^{(n)})] \right. \\ &\quad \left. + [E(X_0) - E(\hat{X}_0^{(n)})])^2\right) \\ &= \sigma^2_{X_0} + \sigma^2_{\hat{X}_0^{(n)}} - 2\mathrm{COV}(X_0, \hat{X}_0^{(n)}) \\ &\quad + [E(X_0) - E(\hat{X}_0^{(n)})]^2 \end{aligned}$$

results in

$$D_X(R) = \lim_{n\to\infty} \left(\sigma^2_{X_0} + (1 - 2a_n)\sigma^2_{\hat{X}_0^{(n)}} + [E(X_0) - E(\hat{X}_0^{(n)})]^2\right). \tag{13.26}$$

Subtracting (13.25) from (13.26) yields

$$\lim_{n\to\infty} \left((1 - a_n)^2 \sigma^2_{\hat{X}_0^{(n)}} + [E(X_0) - E(\hat{X}_0^{(n)})]^2\right) = 0. \tag{13.27}$$

Since both terms in the limit are nonnegative, both must converge to zero since the sum does. Convergence of the rightmost term in the sum proves (13.15). Provided $D_X(R) < \sigma^2_{X_0}$, which is true if $R > 0$, (13.25) and (13.27) together imply that $(a_n - 1)^2/a_n^2$ converges to 0 and hence that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\mathrm{COV}(X_0, \hat{X}_0^{(n)})}{\sigma^2_{\hat{X}_0^{(n)}}} = 1. \tag{13.28}$$

This proves (13.16) and with (13.26) proves (13.17) and also that

$$\lim_{n\to\infty} \mathrm{COV}(X_0, \hat{X}_0^{(n)}) = \sigma^2_{X_0} - D_X(R). \tag{13.29}$$

Finally consider the conditions in terms of the reproduction error. Eq. (13.18) follows from (13.15). Eq. (13.19) follows from (13.15)–(13.29) and some algebra. Eq. (13.20) follows from (13.18) and the asymptotic optimality of the codes. □

If X is a B-process so that $\Delta_{X|Z} = D_X(R)$, then a similar proof yields corresponding results for the simulation problem. If g_n is an asymptotically optimal (with respect to $\overline{\rho}_2$ distortion) sequence of stationary codes of an IID equiprobable source Z with alphabet B of size $R = \log \|B\|$ which produce a simulated process $\tilde{X}^{(n)}$, then

$$\lim_{n\to\infty} E(\tilde{X}_0^{(n)}) = E(X_0)$$
$$\lim_{n\to\infty} \sigma^2_{\tilde{X}_0^{(n)}} = \sigma^2_{X_0} - \Delta_{X|Z}.$$

It is perhaps surprising that when finding the best matching process with constrained rate, the second moments differ.

Finite-order distribution Shannon conditions for IID processes

Several code design algorithms, including randomly populating a trellis to mimic the proof of the trellis source encoding theorem [188], are based on the intuition that the guiding principle of designing such a system for an IID source should be to produce a code with marginal reproduction distribution close to a Shannon optimal reproduction distribution [193, 42, 143]. The following result from [117] formalizes this intuition.

Lemma 13.6. *Given a real-valued IID process X with distribution μ_X, assume that $\{f_n, g_n\}$ is an asymptotically optimal sequence of stationary source encoder/decoder pairs with common alphabet B of size $R = \log \|B\|$ which produce a reproduction process $\hat{X}^{(n)}$. Then a subsequence of the marginal distribution of the reproduction process, $\mu_{\hat{X}_0^{(n)}}$ converges weakly and in quadratic transportation distortion (ρ-bar distortion with respect to squared error distortion) to a Shannon optimal reproduction*

distribution. If the Shannon optimal reproduction distribution is unique, then $\mu_{\hat{X}_0^{(n)}}$ converges to it.

Proof: Given the asymptotically optimal sequence of codes, let π_n denote the induced process joint distributions on $(X, \hat{X}^{(n)})$. The encoded process has alphabet size 2^R and hence entropy rate less than or equal to R. Since coding cannot increase entropy rate, the entropy rate of the reproduction (decoded) process is also less than or equal to R. Since the input process is IID, Lemma 8.8 implies that for all N that

$$\begin{aligned}\frac{1}{N}I(\pi_n^N) &= \frac{1}{N}I(X^N, \hat{X}^N) \ge \frac{1}{N}\sum_{i=0}^{N-1} I(X_i, \hat{X}_i^{(n)}) \\ &= I(X_0, \hat{X}_0^{(n)}) = I(\pi_n^1). \end{aligned} \tag{13.30}$$

The leftmost term converges to the mutual information rate between the input and reproduction, which is bound above by the entropy rate of the output so that

$$I(X_0, \hat{X}_0^{(n)}) \le R, \text{ all } n. \tag{13.31}$$

Since the code sequence is asymptotically optimal, (13.13) holds. Thus the sequence of joint distributions π_n for $(X_0, \hat{X}_0^{(n)})$ meets the conditions of Corollary 9.4 and hence $\mu_{\hat{X}_0^{(n)}}$ has a subsequence which converges weakly to a Shannon optimal distribution. If the Shannon optimal distribution μ_{Y_0} is unique, then every subsequence of of $\mu_{\hat{X}_0^{(n)}}$ has a further subsequence which converges to μ_{Y_0}, which implies that $\mu_{\hat{X}_0^{(n)}}$ converges weakly to μ_{Y_0}. The moment conditions (13.15) and (13.17)) of Lemma 13.5 imply that $E[(\hat{X}_0^{(n)})^2]$ converges to $E[(\hat{X}_0)^2]$. The weak convergence of a subsequence of $\mu_{\hat{X}^{(n)}}$ (or the sequence itself) and the convergence of the second moments imply convergence in quadratic transportation distortion (ρ-bar distortion with respect to squared error distortion) [187]. □

Since the source is IID, the N-fold product of a one-dimensional Shannon optimal distribution is an N-dimensional Shannon optimal distribution. If the Shannon optimal marginal distribution is unique, then so is the N-dimensional Shannon optimal distribution. Since Csiszár's [25] results as summarized in Corollary 9.4 hold for the N-dimensional case, we immediately have the first part of the following corollary.

Corollary 13.1. *Given the assumptions of the lemma, for any positive integer N let $\mu_{\hat{X}^{(n)}}$ denote the N-dimensional joint distribution of the reproduction process $\hat{X}^{(n)}$. Then a subsequence of the N-dimensional reproduction distribution $\mu_{\hat{X}^{(n)}}$ converges weakly and in quadratic transportation distortion to the N-fold product of a Shannon optimal marginal distribution (and hence to an N-dimensional Shannon optimal distribution). If the one dimensional Shannon optimal distribution is unique, then $\mu_{\hat{X}^{(n)}}$ converges*

weakly and in quadratic transportation distortion to its N-fold product distribution.

Proof: The moment conditions (13.15) and (13.17)) of Lemma 13.5 imply that $E[(\hat{X}_k^{(n)})^2]$ converges to $E[(\hat{X}_k)^2]$ for $k = 0, 1, \ldots, N-1$. The weak convergence of the N-dimensional distribution of a subsequence of $\mu_{\hat{X}^{(n)}}$ (or the sequence itself) and the convergence of the second moments imply convergence in quadratic transportation distortion [187]. □

There is no counterpart of this result for optimal codes as opposed to asymptotically optimal codes. Consider the Gaussian case where the Shannon optimal distribution is a product Gaussian distribution with variance $\sigma_X^2 - D_X(R)$. If a code were optimal, then for each N the resulting Nth order reproduction distribution would have to equal the Shannon product distribution. But if this were true for all N, the reproduction would have to be the IID process with the Shannon marginals, but that process has infinite entropy rate.

If X is a B-process, then a small variation on the proof yields similar results for the simulation problem: given an IID target source X, the Nth order joint distributions $\mu_{\tilde{X}^{(n)}}$ of an asymptotically optimal sequence of constrained rate simulations $\tilde{X}^{(n)}$ will have a subsequence that converges weakly and in quadratic transportation distortion to an N-dimensional Shannon optimal distribution.

Asymptotic Uncorrelation

Define as usual the covariance function of the stationary process $\hat{X}^{(n)}$ by $K_{\hat{X}^{(n)}}(k) = \mathrm{COV}(\hat{X}_i^{(n)}, \hat{X}_{i-k}^{(n)})$ for all integer k. The following theorem states and proves an intuitive property nearly optimal codes for IID sources must yield approximately uncorrelated reproduction processes. The result is implied by the convergence of joint distributions to the Shannon optimal distribution along with a technical moment condition proved in the subsequent lemma.

Theorem 13.1. *Given a real-valued IID process X with distribution μ_X, assume that f_n, g_n is an asymptotically optimal sequence of stationary source encoder/decoder pairs with common alphabet B of size $R = \log \|B\|$ which produce a reproduction process $\hat{X}^{(n)}$. For all $k \neq 0$,*

$$\lim_{n\to\infty} K_{\hat{X}^{(n)}}(k) = 0 \tag{13.32}$$

and hence the reproduction processes are asymptotically uncorrelated.

Proof. If the Shannon optimal distribution is unique, then $\mu_{\hat{X}^{(n)}}$ converges in quadratic transportation distortion to the N-fold product of the

Shannon optimal marginal distribution by Corollary 13.1. Lemma 13.7 which follows shows that this implies the convergence of $K_{\hat{X}^{(n)}}(k) = \mathrm{COV}(\hat{X}_k^{(n)}, \hat{X}_0^{(n)})$ to 0 for all $k \neq 0$. □

Taken together these necessary conditions provide straightforward tests for code construction algorithms.

Lemma 13.7. Let μ^N denote the N-fold product of a probability distribution μ on the real line such that $\int x^2 d\mu(x) < \infty$. Assume $\{\nu_n\}$ is a sequence of probability distribution on $\mathbb{R}^N$ such that $\lim_{n\to\infty} \overline{\rho}_N(\mu^N, \nu_n) = 0$. If $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_N^{(n)}$ are random variables with joint distribution ν_n, then for all $i \neq j$, $\lim_{n\to\infty} E[(Y_i^{(n)} - E(Y_i^{(n)}))(Y_j^{(n)} - E(Y_j^{(n)}))] = 0$.

Proof. The convergence of ν_n to μ^N in quadratic transportation distortion implies that there exist IID random variables $Y_1, \ldots, Y_N$ with common distribution μ and a sequence or N random variables $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_N^{(n)}$ with joint distribution ν_n, all defined on the same probability space, such that

$$\lim_{n\to\infty} E[(Y_i^{(n)} - Y_i)^2] = 0, \quad i = 1, \ldots, N. \tag{13.33}$$

First note that this implies for all i

$$\lim_{n\to\infty} E[(Y_i^{(n)})^2] = E[Y_i^2]. \tag{13.34}$$

Since $\lim_{n\to\infty} E|Y_i^{(n)} - Y_i| = 0$, the Cauchy-Scwartz inequality implies that for all i

$$\lim_{n\to\infty} E(Y_i^{(n)}) = E(Y_i). \tag{13.35}$$

The statement is a direct consequence of the fact that in any inner product space, the inner product is jointly continuous. Letting $\langle X, Y\rangle = E(XY)$ and $\|X\| = [E(X^2)]^{1/2}$ for random variables X and Y with finite second moment defined on this probability space, we have the bound

$$\begin{aligned} |\langle Y_i^{(n)}, Y_j^{(n)}\rangle - \langle Y_i, Y_j\rangle| &\le |\langle Y_i^{(n)}, Y_j^{(n)} - Y_j\rangle| + |\langle Y_i^{(n)} - Y_i, Y_j\rangle| \\ &\le \|Y_i^{(n)}\| \, \|Y_j^{(n)} - Y_j\| + \|Y_i^{(n)} - Y_i\| \, \|Y_j\|. \end{aligned}$$

Since $\|Y_i^{(n)}\|$ converges to $\|Y_i\|$ by (13.34) and $\|Y_i^{(n)} - Y_i\|$ converges to zero by (13.33), we obtain that $\langle Y_i^{(n)}, Y_j^{(n)}\rangle$ converges to $\langle Y_i, Y_j\rangle$, i.e,

$$\lim_{n\to\infty} E(Y_i^{(n)} Y_i^{(n)}) = E(Y_i Y_j) = E(Y_i)E(Y_j)$$

since Y_i and Y_j are independent if $i \neq j$. This and (13.35) imply the lemma statement. □

Chapter 14
Coding for Noisy Channels

Abstract Reliable communication over a noisy channel is the focus of this chapter. The chapter begins with a development of the classic fundamental results of Feinstein regarding reliable communication of block codes and the relation of operational channel capacity to Shannon capacity for discrete channels. A technique of Dobrushin is used to extend Feinstein's results for channels with no input memory or anticipation by making codes robust to small changes in the conditional distributions describing channels. This leads in turn to the extension of block coding theorems to d-bar continuous channels, discrete noisy channels where the noise distribution within a block can be well approximated in a d-bar sense with only finite knowledge of past and future inputs. Traditional channel coding theorems for block codes assume knowledge of synchronization — when the blocks begin. Another technique of Dobushin is used to synchronize block codes through noisy channels. Combining synchronized block codes with the Rohlin-Kakutani theorem yields a coding theorem for sliding-block channel coding. Finally, combining the source coding theorems with channel coding theorems yields joint-source and channel coding theorems.

14.1 Noisy Channels

In the treatment of source coding the communication channel was assumed to be noiseless. If the channel is noisy, then the coding strategy must be different, some form of error control is required to undo the damage caused by the channel. The overall point-to-point communication problem is usually broken into two pieces: A source coder is designed for a noiseless channel with a given resolution or rate and an error correction code is designed for the actual noisy channel in order to make it appear almost noiseless. The combination of the two codes

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then provides the desired overall code or joint source and channel code. This division is natural in the sense that optimizing a code for a particular source may suggest quite a different structure than optimizing it for a channel. The structures must be compatible at some point, however, so that they can be used together. This division of source and channel coding is apparent in the subdivision of this chapter. We begin with a fundamental lemma due to Feinstein [39] which is the basis of traditional proofs of coding theorems for channels. It does not consider a source at all, but finds for a given conditional distribution the maximum number of inputs which lead to outputs which can be distinguished with high probability. Feinstein's lemma can be thought of as a channel coding theorem for a channel which is used only once and which has no past or future. The lemma immediately provides a coding theorem for the special case of a channel which has no input memory or anticipation. The difficulties enter when the conditional distributions of output blocks given input blocks depend on previous or future inputs. This difficulty is handled by imposing some form of continuity on the channel with respect to its input, that is, by assuming that if the channel input is known for a big enough block, then the conditional probability of outputs during the same block is known nearly exactly regardless of previous or future inputs. The continuity condition which we shall consider is that of $\overline{d}$-continuous channels. Joint source and channel codes have been obtained for more general channels called *weakly continuous channels* (see, e.g., Kieffer [94] [95]), but these results require a variety of techniques not yet considered here and do not follow as a direct descendent of Feinstein's lemma.

Block codes are extended to sliding-block codes in a manner similar to that for source codes: First it is shown that asynchronous block codes can be synchronized and then that the block codes can be "stationarized" by the insertion of random punctuation. The approach to synchronizing channel codes is based on a technique of Dobrushin [33].

We consider stationary channels almost exclusively, thereby not including interesting nonstationary channels such as finite state channels with an arbitrary starting state. We will discuss such generalizations and we point out that they are straightforward for two-sided processes, but the general theory of AMS channels for one-sided processes is not in a satisfactory state. Lastly, we emphasize ergodic channels. In fact, for the sliding-block codes the channels are also required to be totally ergodic, that is, ergodic with respect to all block shifts.

As previously discussed, we emphasize digital, i.e., discrete, channels. A few of the results, however, are as easily proved under somewhat more general conditions and hence we shall do so. For example, given the background of this book it is actually easier to write things in terms of measures and integrals than in terms of sums over probability mass func-

tions. This additional generality will also permit at least a description of how the results extend to continuous alphabet channels.

14.2 Feinstein's Lemma

Let $(A, \mathcal{B}_A)$ and $(B, \mathcal{B}_B)$ be measurable spaces called the *input space* and the *output space*, respectively. Let P_X denote a probability distribution on $(A, \mathcal{B}_A)$ and let $\nu(F|x)$, $F \in \mathcal{B}_B$, $x \in B$ denote a regular conditional probability distribution on the output space. ν can be thought of as a "channel" with random variables as input and output instead of sequences. Define the hookup $P_X\nu = P_{XY}$ by

$$P_{XY}(F) = \int dP_X(x)\nu(F_x|x).$$

Let P_Y denote the induced output distribution and let $P_X \times P_Y$ denote the resulting product distribution. Assume that $P_{XY} << (P_X \times P_Y)$ and define the Radon-Nikodym derivative

$$f = \frac{dP_{XY}}{d(P_X \times P_Y)} \tag{14.1}$$

and the information density

$$i(x, y) = \ln f(x, y).$$

We use abbreviated notation for densities when the meanings should be clear from context, e.g., f instead of f_{XY}. Observe that for any set F

$$\begin{aligned}\int_F dP_X(x) \left(\int dP_Y(y) f(x, y) \right) &= \int_{F \times B} d(P_X \times P_Y)(x, y) f(x, y) \\ &= \int_{F \times B} dP_{XY}(x, y) = P_X(B) \le 1\end{aligned}$$

and hence

$$\int dP_Y(y) f(x, y) \le 1; \; P_X - \text{a.e.} \tag{14.2}$$

Feinstein's lemma shows that we can pick M inputs $\{x_i \in A; \; i = 1, 2, \ldots, M\}$, and a corresponding collection of M disjoint output events $\{\Gamma_i \in \mathcal{B}_B; \; i = 1, 2, \ldots, M\}$, with the property that given an input x_i with high probability the output will be in Γ_i. We call the collection $C = \{x_i, \Gamma_i; \; i = 1, 2, \ldots, M\}$ a *channel code* or, simply, a code when the meaning is clear from context, with codewords x_i and decoding regions Γ_i. We do not require that the Γ_i exhaust B.

The generalization of Feinstein's original proof for finite alphabets to general measurable spaces is due to Kadota [82] and the following proof is based on his.

Lemma 14.1. *Given an integer M and $a > 0$ there exist $x_i \in A$; $i = 1, \ldots, M$ and a measurable partition $\mathcal{F} = \{\Gamma_i;\ i = 1, \ldots, M\}$ of B such that*

$$\nu(\Gamma_i^c | x_i) \leq Me^{-a} + P_{XY}(i \leq a).$$

Proof: Define $G = \{x, y : i(x, y) > a\}$ Set $\epsilon = Me^{-a} + P_{XY}(i \leq a) = Me^{-a} + P_{XY}(G^c)$. The result is obvious if $\epsilon \geq 1$ and hence we assume that $\epsilon < 1$ and hence also that

$$P_{XY}(G^c) \leq \epsilon < 1$$

and therefore that

$$P_{XY}(i > a) = P_{XY}(G) = \int dP_X(x)\nu(G_x|x) > 1 - \epsilon > 0.$$

This implies that the set $\tilde{A} = \{x : \nu(G_x|x) > 1 - \epsilon$ and (14.2) holds$\}$ must have positive measure under P_X We now construct a code consisting of input points x_i and output sets Γ_{x_i}. Choose an $x_1 \in \tilde{A}$ and define $\Gamma_{x_1} = G_{x_1}$. Next choose if possible a point $x_2 \in \tilde{A}$ for which $\nu(G_{x_2} - \Gamma_{x_1}|x_2) > 1 - \epsilon$. Continue in this way until either M points have been selected or all the points in $\tilde{A}$ have been exhausted. In particular, given the pairs $\{x_j, \Gamma_j\}$; $j = 1, 2, \ldots, i-1$, satisfying the condition, find an x_i for which

$$\nu(G_{x_i} - \bigcup_{j<i} \Gamma_{x_j} | x_i) > 1 - \epsilon. \tag{14.3}$$

If the procedure terminates before M points have been collected, denote the final point's index by n. Observe that

$$\nu({\Gamma_{x_i}}^c | x_i) \leq \nu({G_{x_i}}^c | x_i) \leq \epsilon;\ i = 1, 2, \ldots, n$$

and hence the lemma will be proved if we can show that necessarily n cannot be strictly less than M. We do this by assuming the contrary and finding a contradiction.

Suppose that the selection has terminated at $n < M$ and define the set $F = \bigcup_{i=1}^n \Gamma_{x_i} \in \mathcal{B}_B$. Consider the probability

$$P_{XY}(G) = P_{XY}(G \bigcap (A \times F)) + P_{XY}(G \bigcap (A \times F^c)). \tag{14.4}$$

The first term can be bounded above as

$$P_{XY}(G \bigcap (A \times F)) \leq P_{XY}(A \times F) = P_Y(F) = \sum_{i=1}^n P_Y(\Gamma_{x_i}).$$

We also have from the definitions and from (14.2) that

$$P_Y(\Gamma_{x_i}) = \int_{\Gamma_{x_i}} dP_Y(y) \le \int_{G_{x_i}} dP_Y(y) \le \int_{G_{x_i}} \frac{f(x_i, y)}{e^a} dP_Y(y)$$
$$\le e^{-a} \int dP_Y(y) f(x_i, y) \le e^{-a}$$

and hence

$$P_{XY}(G \bigcap (A \times F)) \le n e^{-a}. \tag{14.5}$$

Consider the second term of (14.3):

$$\begin{aligned} P_{XY}(G \bigcap (A \times F^c)) &= \int dP_X(x) \nu((G \bigcap (A \times F^c))_x | x) \\ &= \int dP_X(x) \nu(G_x \bigcap F^c | x) \\ &= \int dP_X(x) \nu(G_x - \bigcup_{i=1}^{n} \Gamma_i | x). \end{aligned} \tag{14.6}$$

We must have, however, that

$$\nu(G_x - \bigcup_{i=1}^{n} \Gamma_i | x) \le 1 - \epsilon$$

with P_X probability 1 or there would be a point x_{n+1} for which

$$\nu(G_{x_{n+1}} - \bigcup_{i=1}^{n+1} \Gamma_i | x_{n+1}) > 1 - \epsilon,$$

that is, (14.3) would hold for $i = n + 1$, contradicting the definition of n as the largest integer for which (14.3) holds. Applying this observation to (14.6) yields

$$P_{XY}(G \bigcap (A \times F^c)) \le 1 - \epsilon$$

which with (14.4) and (14.5) implies that

$$P_{XY}(G) \le n e^{-a} + 1 - \epsilon. \tag{14.7}$$

From the definition of ϵ, however, we have also that

$$P_{XY}(G) = 1 - P_{XY}(G^c) = 1 - \epsilon + M e^{-a}$$

which with (14.7) implies that $M \le n$, completing the proof. □

14.3 Feinstein's Theorem

Given a channel $[A, \nu, B]$ an (M, n, ϵ) block channel code for ν is a collection $\{w_i, \Gamma_i\}$; $i = 1, 2, \ldots, M$, where $w_i \in A^n$, $\Gamma_i \in \mathcal{B}_B^n$, all i, with the property that

$$\sup_{x \in c(w_i)} \max_{i=1,\ldots,M} \nu_x^n(\Gamma_i) \le \epsilon, \tag{14.8}$$

where $c(a^n) = \{x : x^n = a^n\}$ and where ν_x^n is the restriction of ν_x to $\mathcal{B}_B^n$. The rate of the code is defined as $n^{-1} \log M$. Thus an (n, M, ϵ) channel code is a collection of M input n-tuples and corresponding output cells such that regardless of the past or future inputs, if the input during time 1 to n is a channel codeword, then the output during time 1 to n is very likely to lie in the corresponding output cell. Channel codes will be useful in a communication system because they permit nearly error free communication of a select group of messages or codewords. A communication system can then be constructed for communicating a source over the channel reliably by mapping source blocks into channel codewords. If there are enough channel codewords to assign to all of the source blocks (at least the most probable ones), then that source can be reliably reproduced by the receiver. Hence a fundamental issue for such an application will be the number of messages M or, equivalently, the rate R of a channel code.

Feinstein's lemma can be applied fairly easily to obtain something that resembles a coding theorem for a noisy channel. Suppose that $[A, \nu, B]$ is a channel and $[A, \mu]$ is a source and that $[A \times B, p = \mu\nu]$ is the resulting hookup. Denote the resulting pair process by $\{X_n, Y_n\}$ For any integer K let p^K denote the restriction of p to $(A^K \times B^K, \mathcal{B}_A^K \times \mathcal{B}_B^K)$, that is, the distribution on input/output K-tuples (X^K, Y^K). The joint distribution p^K together with the input distribution μ^K induce a regular conditional probability $\hat{\nu}^K$ defined by $\hat{\nu}^K(F|x^K) = \Pr(Y^K \in F | X^K = x^K)$. In particular,

$$\begin{aligned} \hat{\nu}^K(G|a^K) &= \Pr(Y^K \in G | X^K = a^K) \\ &= \frac{1}{\mu^K(a^K)} \int_{c(a^K)} \nu_x^K(G) d\mu(x). \end{aligned} \tag{14.9}$$

where $c(a^K) = \{x : x^K = a^K\}$ is the rectangle of all sequences with a common K-dimensional output. We call $\hat{\nu}^K$ the *induced K-dimensional channel* of the channel ν and the source μ. It is important to note that the induced channel depends on the source as well as on the channel, a fact that will cause some difficulty in applying Feinstein's lemma. An exception to this case which proves to be an easy application is that of a channel without input memory and anticipation, in which case we have from the definitions that

$$\hat{\nu}^K(F|a^K) = \nu_x(Y^K \in F);\ x \in c(a^K).$$

Application of Feinstein's lemma to the induced channel yields the following result, which was proved by Feinstein for stationary finite alphabet channels and is known as Feinstein's theorem:

Lemma 14.2. *Suppose that $[A \times B, \mu\nu]$ is an AMS and ergodic hookup of a source μ and channel ν. Let $\bar{I}_{\mu\nu} = \bar{I}_{\mu\nu}(X;Y)$ denote the average mutual information rate and assume that $\bar{I}_{\mu\nu} = I^*_{\mu\nu}$ is finite (as is the case if the alphabets are finite (Theorem 8.2) or have the finite-gap information property (Theorem 8.4)). Then for any $R < \bar{I}_{\mu\nu}$ and any $\epsilon > 0$ there exists for sufficiently large n a code $\{w_i^n; \Gamma_i;\ i = 1, 2, \ldots, M\}$, where $M = \lfloor e^{nR} \rfloor$, $w_i^n \in A^n$, and $\Gamma_i \in \mathcal{B}_B^n$, with the property that*

$$\hat{\nu}^n(\Gamma_i^c | w_i^n) \le \epsilon, i = 1, 2, \ldots, M. \tag{14.10}$$

Comment: We shall call a code $\{w_i, \Gamma_i;\ i = 1, 2, \ldots, M\}$ which satisfies (14.10) for a channel input process μ a (μ, M, n, ϵ)-Feinstein code. The quantity $n^{-1} \log M$ is called the *rate* of the Feinstein code.

Proof: Let η denote the output distribution induced by μ and ν. Define the information density

$$i_n = \frac{dp^n}{d(\mu^n \times \eta^n)}$$

and define

$$\delta = \frac{\bar{I}_{\mu\nu} - R}{2} > 0.$$

Apply Feinstein's lemma to the n-dimensional hookup $(\mu\nu)^n$ with $M = \lfloor e^{nR} \rfloor$ and $a = n(R + \delta)$ to obtain a code $\{w_i, \Gamma_i\};\ i = 1, 2, \ldots, M$ with

$$\begin{aligned} &\max_i \hat{\nu}^n(\Gamma_i^c | w_i^n) \\ &\le M e^{-n(R+\delta)} + p^n(i_n \le n(R + \delta)) \\ &= \lfloor e^{nR} \rfloor e^{-n(R+\delta)} + p(\frac{1}{n} i_n(X^n; Y^n) \le R + \delta) \end{aligned} \tag{14.11}$$

and hence

$$\max_i \hat{\nu}^n(\Gamma_i^c | w_i^n) \le e^{-n\delta} + p(\frac{1}{n} i_n(X^n; Y^n) \le \bar{I}_{\mu\nu} - \delta). \tag{14.12}$$

From Theorem 8.1 $n^{-1} i_n$ converges in L^1 to $\bar{I}_{\mu\nu}$ and hence it also converges in probability. Thus given ϵ we can choose an n large enough to ensure that the right hand side of (14.11) is smaller than ϵ, which completes the proof of the theorem. □

We said that the lemma "resembled" a coding theorem because a real coding theorem would prove the existence of an (M, n, ϵ) channel code, that is, it would concern the channel ν itself and not the induced channel $\hat{\nu}$, which depends on a channel input process distribution μ. The difference between a Feinstein code and a channel code is that the Feinstein code has a similar property for an induced channel which in general depends on a source distribution, while the channel code has this property independent of any source distribution and for any past or future inputs.

Feinstein codes will be used to construct block codes for noisy channels. The simplest such construction is presented next.

Corollary 14.1. *Suppose that a channel $[A, \nu, B]$ is input memoryless and input nonanticipatory as defined in Section 2.9. Then a (μ, M, n, ϵ)-Feinstein code for some channel input process μ is also an (M, n, ϵ)-code.*

Proof: Immediate since for a channel without input memory and anticipation we have that $\nu_x^n(F) = \nu_u^n(F)$ if $x^n = u^n$. □

The principal idea of constructing channel codes from Feinstein codes for more general channels will be to place assumptions on the channel which ensure that for sufficiently large n the channel distribution ν_x^n and the induced finite dimensional channel $\hat{\nu}^n(\cdot|x^n)$ are close. This general idea was proposed by McMillan [123] who suggested that coding theorems would follow for channels that were sufficiently continuous in a suitable sense.

The previous results did not require stationarity of the channel, but in a sense stationarity is implicit if the channel codes are to be used repeatedly (as they will be in a communication system). Thus the immediate applications of the Feinstein results will be to stationary channels.

The following is a rephrasing of Feinstein's theorem that will be useful.

Corollary 14.2. *Suppose that $[A \times B, \mu\nu]$ is an AMS and ergodic hookup of a source μ and channel ν. Let $\bar{I}_{\mu\nu} = \bar{I}_{\mu\nu}(X;Y)$ denote the average mutual information rate and assume that $\bar{I}_{\mu\nu} = I^*_{\mu\nu}$ is finite. Then for any $R < \bar{I}_{\mu\nu}$ and any $\epsilon > 0$ there exists an n_0 such that for all $n \ge n_0$ there are $(\mu, \lfloor e^{nR} \rfloor, n, \epsilon)$-Feinstein codes.*

As a final result of the Feinstein variety, we point out a variation that applies to nonergodic channels.

Corollary 14.3. *Suppose that $[A \times B, \mu\nu]$ is an AMS hookup of a source μ and channel ν. Suppose also that the information density converges a.e. to a limiting density*

$$i_\infty = \lim_{n\to\infty} \frac{1}{n} i_n(X^n; Y^n).$$

(Conditions for this to hold are given in Theorem 11.4.) Then given $\epsilon > 0$ and $\delta > 0$ there exists for sufficiently large n a $[\mu, M, n, \epsilon + \mu\nu(i_\infty \le R + \delta)]$ Feinstein code with $M = \lfloor e^{nR} \rfloor$.

Proof: Follows from the lemma and from Fatou's lemma which implies that

$$\limsup_{n\to\infty} p(\frac{1}{n} i_n(X^n;Y^n) \le a) \le p(i_\infty \le a).$$

□

14.4 Channel Capacity

The form of the Feinstein lemma and its corollaries invites the question of how large R (and hence M) can be made while still getting a code of the desired form. From Feinstein's theorem it is seen that for an ergodic channel R can be any number less than $\overline{I}(\mu\nu)$ which suggests that if we define the quantity

$$C_{\text{AMS,e}} = \sup_{\text{AMS and ergodic } \mu} \overline{I}_{\mu\nu}, \tag{14.13}$$

then if $\overline{I}_{\mu\nu} = I^*_{\mu\nu}$ (e.g., the channel has finite alphabet), then we can construct for some μ a Feinstein code for μ with rate R arbitrarily near $C_{\text{AMS,e}}$. $C_{\text{AMS,e}}$ is an example of a quantity called an *information rate capacity* or, simply, *capacity* of a channel. We shall encounter a few variations on this definition just as there were various ways of defining distortion-rate functions for sources by considering either vectors or processes with different constraints. In this section a few of these definitions are introduced and compared.

A few possible definitions of information rate capacity are

$$C_{\text{AMS}} = \sup_{\text{AMS } \mu} \overline{I}_{\mu\nu}, \tag{14.14}$$

$$C_{\text{s}} = \sup_{\text{stationary } \mu} \overline{I}_{\mu\nu}, \tag{14.15}$$

$$C_{\text{s,e}} = \sup_{\text{stationary and ergodic } \mu} \overline{I}_{\mu\nu}, \tag{14.16}$$

$$C_{\text{ns}} = \sup_{n-\text{stationary } \mu} \overline{I}_{\mu\nu}, \tag{14.17}$$

$$C_{\text{bs}} = \sup_{\text{block stationary } \mu} \overline{I}_{\mu\nu} = \sup_n \sup_{n-\text{stationary } \mu} \overline{I}_{\mu\nu}. \tag{14.18}$$

Several inequalities are obvious from the definitions:

$$C_{\text{AMS}} \ge C_{\text{bs}} \ge C_{\text{ns}} \ge C_{\text{s}} \ge C_{\text{s,e}} \tag{14.19}$$

$$C_{\text{AMS}} \ge C_{\text{AMS,e}} \ge C_{\text{s,e}}. \tag{14.20}$$

In order to relate these definitions we need a variation on Lemma 12.3.1 described in the following lemma.

Lemma 14.3. *Given a stationary finite-alphabet channel* $[A, \nu, B]$*, let* μ *be the distribution of a stationary channel input process and let* $\{\mu_x\}$ *be its ergodic decomposition. Then*

$$\bar{I}_{\mu\nu} = \int d\mu(x)\bar{I}_{\mu_x\nu}. \tag{14.21}$$

Proof: We can write

$$\bar{I}_{\mu\nu} = h_1(\mu) - h_2(\mu)$$

where

$$h_1(\mu) = \overline{H}_\eta(Y) = \inf_n \frac{1}{n} H_\eta(Y^n)$$

is the entropy rate of the output, where η is the output measure induced by μ and ν, and where

$$h_2(\mu) = \overline{H}_{\mu\nu}(Y|X) = \lim_{n\to\infty} \frac{1}{n} H_{\mu\nu}(Y^n|X^n)$$

is the conditional entropy rate of the output given the input. If $\mu_k \to \mu$ on any finite dimensional rectangle, then also $\eta_k \to \eta$ and hence $H_{\eta_k}(Y^n) \to H_\eta(Y^n)$ so that it follows as in the proof of Corollary 3.4 that $h_1(\mu)$ is an upper semicontinuous function of μ. It is also affine because $\overline{H}_\eta(Y)$ is an affine function of η (Lemma 3.9) which is in turn a linear function of μ. Thus from Theorem 8.9.1 of [55] or Theorem 8.5 of [58]

$$h_1(\mu) = \int d\mu(x) h_1(\mu_x).$$

$h_2(\mu)$ is also affine in μ since $h_1(\mu)$ is affine in μ and $\bar{I}_{\mu\nu}$ is affine in μ (since it is affine in $\mu\nu$ from Lemma 8.6). Hence we will be done if we can show that $h_2(\mu)$ is upper semicontinuous in μ since then Theorem 8.9.1 of [55] will imply that

$$h_2(\mu) = \int d\mu(x) h_2(\mu_x)$$

which with the corresponding result for h_1 proves the lemma. To see this observe that if $\mu_k \to \mu$ on finite dimensional rectangles, then

$$H_{\mu_k\nu}(Y^n|X^n) \to H_{\mu\nu}(Y^n|X^n). \tag{14.22}$$

Next observe that for stationary processes

$$\begin{aligned} H(Y^n|X^n) &\le H(Y^m|X^n) + H(Y_m^{n-m}|X^n) \\ &\le H(Y^m|X^m) + H(Y_m^{n-m}|X_m^{n-m}) \\ &= H(Y^m|X^m) + H(Y^{n-m}|X^{n-m}) \end{aligned}$$

which as in Section 2.4 implies that $H(Y^n|X^n)$ is a subadditive sequence and hence

$$\lim_{n\to\infty} \frac{1}{n} H(Y^n|X^n) = \inf_n \frac{1}{n} H(Y^n|X^n).$$

Coupling this with (14.22) proves upper semicontinuity exactly as in the proof of Corollary 3.4, which completes the proof of the lemma. □

Lemma 14.4. *If a channel ν has a finite alphabet and is stationary, then all of the above information rate capacities are equal.*

Proof: From Theorem 8.2 $\overline{I} = I^*$ for finite alphabet processes and hence from Lemma 8.6 and Lemma 2.2 we have that if μ is AMS with stationary mean $\overline{\mu}$, then

$$\overline{I}_{\mu\nu} = \overline{I}_{\overline{\mu}\nu} = \overline{I}_{\overline{\mu}\overline{\nu}}$$

and thus the supremum over AMS sources must be the same as that over stationary sources. The fact that $C_{\rm s} \le C_{\rm s,e}$ follows immediately from the previous lemma since the best stationary source can do no better than to put all of its measure on the ergodic component yielding the maximum information rate. Combining these facts with (14.19)–(14.20) proves the lemma. □

Because of the equivalence of the various forms of information rate capacity for stationary channels, we shall use the symbol C to represent the information rate capacity of a stationary channel and observe that it can be considered as the solution to any of the above maximization problems.

Shannon's original definition of channel capacity applied to channels without input memory or anticipation. We pause to relate this definition to the process definitions. Suppose that a channel $[A, \nu, B]$ has no input memory or anticipation and hence for each n there are regular conditional probability measures $\hat{\nu}^n(G|x^n)$; $x \in A^n$, $G \in \mathcal{B}_B^n$, such that

$$\nu_x^n(G) = \hat{\nu}^n(G|x^n).$$

Define the finite-dimensional capacity of the $\hat{\nu}^n$ by

$$C_n(\hat{\nu}^n) = \sup_{\mu^n} I_{\mu^n\hat{\nu}^n}(X^n; Y^n),$$

where the supremum is over all vector distributions μ^n on A^n. Define the Shannon capacity of the channel μ by

$$C_{\rm Shannon} = \lim_{n\to\infty} \frac{1}{n} C^n(\hat{\nu}^n)$$

if the limit exists. Suppose that the Shannon capacity exists for a channel ν without memory or anticipation. Choose N large enough so that C_N is very close to C_{Shannon} and let μ^N approximately yield C_N. Then construct a block memoryless source using μ^N. A block memoryless source is AMS and hence if the channel is AMS we must have an information rate

$$\bar{I}_{\mu\nu}(X;Y) = \lim_{n\to\infty} \frac{1}{n} I_{\mu\nu}(X^n;Y^n) = \lim_{k\to\infty} \frac{1}{kN} I_{\mu\nu}(X^{kN};Y^{kN}).$$

Since the input process is block memoryless, we have from Lemma 8.8 that

$$I(X^{kN};Y^{kN}) \ge \sum_{i=0}^{k} I(X_{iN}^N;Y_{iN}^N).$$

If the channel is stationary then $\{X_n, Y_n\}$ is N-stationary and hence if

$$\frac{1}{N} I_{\mu^N\hat{\nu}^N}(X^N;Y^N) \ge C_{\text{Shannon}} - \epsilon,$$

then

$$\frac{1}{kN} I(X^{kN};Y^{kN}) \ge C_{\text{Shannon}} - \epsilon.$$

Taking the limit as $k \to \infty$ we have that

$$C_{\text{AMS}} = C \ge \bar{I}(X;Y) = \lim_{k\to\infty} \frac{1}{kN} I(X^{kN};Y^{kN}) \ge C_{\text{Shannon}} - \epsilon$$

and hence

$$C \ge C_{\text{Shannon}}.$$

Conversely, pick a stationary source μ which nearly yields $C = C_{\text{s}}$, that is,

$$\bar{I}_{\mu\nu} \ge C_{\text{s}} - \epsilon.$$

Choose n_0 sufficiently large to ensure that

$$\frac{1}{n} I_{\mu\nu}(X^n;Y^n) \ge \bar{I}_{\mu\nu} - \epsilon \ge C_{\text{s}} - 2\epsilon.$$

This implies, however, that for $n \ge n_0$

$$C_n \ge C_{\text{s}} - 2\epsilon,$$

and hence application of the previous lemma proves the following lemma.

Lemma 14.5. *Given a finite alphabet stationary channel ν with no input memory or anticipation,*

$$C = C_{\text{AMS}} = C_{\text{s}} = C_{\text{s,e}} = C_{\text{Shannon}}.$$

The Shannon capacity is of interest because it can be numerically computed while the process definitions are not always amenable to such computation.

With Corollary 14.2 and the definition of channel capacity we have the following result.

Lemma 14.6. *If ν is an AMS and ergodic channel and $R < C$, then there is an n_0 sufficiently large to ensure that for all $n \geq n_0$ there exist $(\mu, \lfloor e^{nR} \rfloor, n, \epsilon)$ Feinstein codes for some channel input process μ.*

Corollary 14.4. *Suppose that $[A, \nu, B]$ is an AMS and ergodic channel with no input memory or anticipation. Then if $R < C$, the information rate capacity or Shannon capacity, then for $\epsilon > 0$ there exists for sufficiently large n a $(\lfloor e^{nR} \rfloor, n, \epsilon)$ channel code.*

Proof: Follows immediately from Corollary 14.3 by choosing a stationary and ergodic source μ with $\overline{I}_{\mu\nu} \in (R, C)$. □

There is another, quite different, notion of channel capacity that we introduce for comparison and to aid the discussion of nonergodic stationary channels. Define for an AMS channel ν and any $\lambda \in (0, 1)$ the quantile

$$C^*(\lambda) = \sup_{\text{AMS } \mu} \sup\{r : \mu\nu(i_\infty \leq r) < \lambda)\},$$

where the supremum is over all AMS channel input processes and i_∞ is the limiting information density (which exists because $\mu\nu$ is AMS and has finite alphabet). Define the *information quantile capacity* C^* by

$$C^* = \lim_{\lambda \to 0} C^*(\lambda).$$

The limit is well-defined since the $C^*(\lambda)$ are bounded and nonincreasing. The information quantile capacity was introduced by Winkelbauer [194] and its properties were developed by him and by Kieffer [90]. Fix an $R < C^*$ and define $\delta = (C^* - R)/2$. Given $\epsilon > 0$ we can find from the definition of C^* an AMS channel input process μ for which $\mu\nu(i_\infty \leq R + \delta) \leq \epsilon$. Applying Corollary 14.3 with this δ and $\epsilon/2$ then yields the following result for nonergodic channels.

Lemma 14.7. *If ν is an AMS channel and $R < C^*$, then there is an n_0 sufficiently large to ensure that for all $n \geq n_0$ there exist $(\mu, fe^{nR}f, n, \epsilon)$ Feinstein codes for some channel input process μ.*

We close this section by relating C and C^* for AMS channels.

Lemma 14.8. *Given an AMS channel ν,*

$$C \geq C^*.$$

Proof: Fix $\lambda > 0$. If $r < C^*(\lambda)$ there is a μ such that $\lambda > \mu\nu(i_\infty \le r) = 1 - \mu\nu(i_\infty > r) \ge 1 - \bar{I}_{\mu\nu}/r$, where we have used the Markov inequality. Thus for all $r < C^*$ we have that $\bar{I}_{\mu\nu} \ge r(1 - \mu\nu(i_\infty \le r))$ and hence

$$C \ge \bar{I}_{\mu\nu} \ge C^*(\lambda)(1-\lambda) \underset{\lambda \to 0}{\to} C^*.$$

□

It can be shown that if a stationary channel is also ergodic, then $C = C^*$ by using the ergodic decomposition to show that the supremum defining $C(\lambda)$ can be taken over ergodic sources and then using the fact that for ergodic μ and ν, i_∞ equals $\bar{I}_{\mu\nu}$ with probability one. (See Kieffer [90].)

14.5 Robust Block Codes

Feinstein codes immediately yield channel codes when the channel has no input memory or anticipation because the induced vector channel is the same with respect to vectors as the original channel. When extending this technique to channels with memory and anticipation we will try to ensure that the induced channels are still reasonable approximations to the original channel, but the approximations will not be exact and hence the conditional distributions considered in the Feinstein construction will not be the same as the channel conditional distributions. In other words, the Feinstein construction guarantees a code that works well for a conditional distribution formed by averaging the channel over its past and future using a channel input distribution that approximately yields channel capacity. This does not in general imply that the code will also work well when used on the unaveraged channel with a particular past and future input sequence. We solve this problem by considering channels for which the two distributions are close if the block length is long enough.

In order to use the Feinstein construction for one distribution on an actual channel, we will modify the block codes slightly so as to make them *robust* in the sense that if they are used on channels with slightly different conditional distributions, their performance as measured by probability of error does not change much. In this section we prove that this can be done. The basic technique is due to Dobrushin [33] and a similar technique was studied by Ahlswede and Gács [4]. (See also Ahlswede and Wolfowitz [5].) The results of this section are due to Gray, Ornstein, and Dobrushin [68].

A channel block length n code $\{w_i, \Gamma_i;\ i = 1, 2, \ldots, M$ will be called δ-*robust* (in the Hamming distance sense) if the decoding sets Γ_i are such that the expanded sets

$$(\Gamma_i)_\delta \equiv \{y^n : \frac{1}{n} d_n(y^n, \Gamma_i) \le \delta\}$$

are disjoint, where

$$d_n(y^n, \Gamma_i) = \min_{u^n \in \Gamma_i} d_n(y^n, u^n)$$

and

$$d_n(y^n, u^n) = \sum_{i=0}^{n-1} d_H(y_i, u_i)$$

and $d_H(a,b)$ is the Hamming distance (1 if $a \neq b$ and 0 if $a = b$). Thus the code is δ-robust if received n-tuples in a decoding set can be changed by an average Hamming distance of up to δ without falling in a different decoding set. We show that by reducing the rate of a code slightly we can always make a Feinstein code robust.

Lemma 14.9. *Let* $\{w_i', \Gamma_i';\ i = 1, 2, \ldots, M'\}$ *be a* $(\mu, e^{nR'}, n, \epsilon)$*-Feinstein code for a channel* ν*. Given* $\delta \in (0, 1/4)$ *and*

$$R < R' - h_2(2\delta) - 2\delta \log(\|B\| - 1),$$

where as before $h_2(a)$ *is the binary entropy function* $-a \log a - (1 - a) \log(1 - a)$*, there exists a* δ*-robust* $(\mu, \lfloor e^{nR} \rfloor, n, \epsilon_n)$*-Feinstein code for* ν *with*

$$\epsilon_n \le \epsilon + e^{-n(R' - R - h_2(2\delta) - 2\delta \log(\|B\| - 1) - 3/n)}.$$

Proof: For $i = 1, 2, \ldots, M'$ let $r_i(y^n)$ denote the indicator function for $(\Gamma_i)_{2\delta}$. For a fixed y^n there can be at most

$$\sum_{i=0}^{2\delta n} \binom{n}{i} (\|B\| - 1)^i = \|B\|^n \sum_{i=0}^{2\delta n} \binom{n}{i} (1 - \frac{1}{\|B\|})^i \frac{(1}{\|B\|})^{n-i}$$

n-tuples $b^n \in B^n$ such that $n^{-1} d_n(y^n, b^n) \le 2\delta$. Set $p = 1 - 1/\|B\|$ and apply Lemma 3.6 to the sum to obtain the bound

$$\|B\|^n \sum_{i=0}^{2\delta n} \binom{n}{k} (1 - \frac{1}{\|B\|})^i (\frac{1}{\|B\|})^{n-i} \le \|B\|^n e^{-n h_2(2\delta \| p)}$$
$$= e^{-n h_2(2\delta \| p) + n \log \|B\|},$$

where

$$\begin{aligned} h_2(2\delta\|p) &= 2\delta\ln\frac{2\delta}{p} + (1-2\delta)\ln\frac{1-2\delta}{1-p} \\ &= -h_2(\delta) + 2\delta\ln\frac{\|B\|}{\|B\|-1} + (1-2\delta)\ln\|B\| \\ &= -h_2(\delta) + \ln\|B\| - 2\delta\ln(\|B\|-1). \end{aligned}$$

Combining this bound with the fact that the Γ_i are disjoint we have that

$$\sum_{i=1}^{M'} r_i(y^n) \le \sum_{i=0}^{2\delta n}\binom{n}{i}(\|B\|-1)^i \le e^{-n(h_2(2\delta)+2\delta\ln(\|B\|-1)}.$$

Set $M = \lfloor e^{nR}\rfloor$ and select $2M$ subscripts $k_1,\cdots,k_{2M}$ from $\{1,\cdots,M'\}$ by random equally likely independent selection without replacement so that each index pair $(k_j,k_m);\ j,m = 1,\ldots,2M;\ j\ne m$, assumes any unequal pair with probability $(M'(M'-1))^{-1}$. We then have that

$$\begin{aligned} &E\left(\frac{1}{2M}\sum_{j=1}^{2M}\sum_{m=1,m\ne j}^{2M}\hat{\nu}(\Gamma'_{k_j}\bigcap(\Gamma'_{k_m})_{2\delta}|w'_{k_j})\right) \\ &= \frac{1}{2M}\sum_{j=1}^{2M}\sum_{m=1,m\ne j}^{2M}\sum_{k=1}^{M'}\sum_{i=1,i\ne k}^{M'}\frac{1}{M'(M'-1)}\sum_{y^n\in\Gamma'_k}\hat{\nu}(y^n|w'_k)r_i(y^n) \\ &\le \frac{1}{2M}\sum_{j=1}^{2M}\sum_{m=1,m\ne j}^{2M}\sum_{k=1}^{M'}\frac{1}{M'(M'-1)}\sum_{y^n\in\Gamma'_k}\hat{\nu}(y^n|w'_k)\sum_{i=1,i\ne k}^{M'}r_i(y^n) \\ &\le \frac{2M}{M-1}e^{n(h_2(2\delta)+2\delta\log(\|B\|-1)} \\ &\le 4e^{-n(R'-R-h_2(2\delta)-2\delta\log(\|B\|-1)} \equiv \lambda_n, \end{aligned}$$

where we have assumed that $M'\ge 2$ so that $M'-1\ge M'/2$. Analogous to a random coding argument, since the above expectation is less than λ_n, there must exist a fixed collection of subscripts $i_1,\cdots,i_{2M'}$ such that

$$\frac{1}{2M}\sum_{j=1}^{2M}\sum_{m=1,m\ne j}^{2M}\hat{\nu}(\Gamma'_{i_j}\bigcap(\Gamma'_{i_m})_{2\delta}|w_{i'_j}) \le \lambda_n.$$

Since no more than half of the above indices can exceed twice the expected value, there must exist indices $k_1,\cdots,k_M \in \{j_1,\cdots,j_{2M}\}$ for which

$$\sum_{m=1,m\ne j}^{M}\hat{\nu}(\Gamma'_{k_j}\bigcap(\Gamma'_{k_m})_{2\delta}|w'_{k_j}) \le 2\lambda_n;\ i = 1,2,\ldots,M.$$

Define the code $\{w_i,\Gamma_i;\ i=1,\ldots,M\}$ by $w_i = w'_{k_i}$ and

$$\Gamma_i = \Gamma'_{k_i} - \bigcup_{m=1, m\neq i}^{M'} (\Gamma'_{k_m})_{2\delta}.$$

The $(\Gamma_i)_\delta$ are obviously disjoint since we have removed from Γ'_{k_i} all words within 2δ of a word in any other decoding set. Furthermore, we have for all $i = 1, 2, \ldots, M$ that

$$\begin{aligned}
1 - \epsilon &\le \hat{\nu}(\Gamma'_{k_i} | w'_{k_i}) \\
&= \hat{\nu}(\Gamma'_{k_i} \bigcap \left(\bigcup_{m\neq i} (\Gamma'_{k_m})_{2\delta} \right) | w'_{k_i}) + \hat{\nu}(\Gamma'_{k_i} \bigcap \left(\bigcup_{m\neq i} (\Gamma'_{k_m})_{2\delta} \right)^c | w'_{k_i}) \\
&\le \sum_{m\neq i} \hat{\nu}(\Gamma'_{k_i} \bigcap (\Gamma'_{k_m})_{2\delta} | w'_{k_i}) + \hat{\nu}(\Gamma_i | w_i) \\
&< 2\lambda_n + \hat{\nu}(\Gamma_i | w_i)
\end{aligned}$$

and hence

$$\hat{\nu}(\Gamma_i | w_i) \ge 1 - \epsilon - 8e^{-n(R' - R - h_2(2\delta) - 2\delta \log(\|B\| - 1)},$$

which proves the lemma. □

Corollary 14.5. *Let ν be a stationary channel and let C_n be a sequence of $(\mu_n, \lfloor e^{nR'} \rfloor, n, \epsilon/2)$ Feinstein codes for $n \ge n_0$. Given an $R > 0$ and $\delta > 0$ such that $R < R' - h_2(2\delta) - 2\delta \log(\|B\| - 1)$, there exists for n_1 sufficiently large a sequence C'_n; $n \ge n_1$, of δ-robust $(\mu_n, \lfloor e^{nR} \rfloor, n, \epsilon)$ Feinstein codes.*

Proof: The corollary follows from the lemma by choosing n_1 so that

$$e^{-n_1(R' - R - h_2(2\delta) - 2\delta \ln(\|B\| - 1) - 3/n_1)} \le \frac{\epsilon}{2}.$$

□

Note that the sources may be different for each n and that n_1 does not depend on the channel input measure.

14.6 Block Coding Theorems for Noisy Channels

Suppose now that ν is a stationary finite alphabet $\overline{d}$-continuous channel. Suppose also that for $n \ge n_1$ we have a sequence of δ-robust $(\mu_n, \lfloor e^{nR} \rfloor, n, \epsilon)$ Feinstein codes $\{w_i, \Gamma_i\}$ as in the previous section. We now quantify the performance of these codes when used as channel block codes, that is, used on the actual channel ν instead of on an induced channel. As previously let $\hat{\nu}^n$ be the n-dimensional channel induced by μ_n and the channel ν, that is, for $\mu_n^n(a^n) > 0$

$$\hat{\nu}^n(G|a^n) = \Pr(Y^n \in G | X^n = a^n) = \frac{1}{\mu_n^n(a^n)} \int_{c(a^n)} \nu_x^n(G)\, d\mu(x), \tag{14.23}$$

where $c(a^n)$ is the rectangle $\{x : x \in A^{\mathbb{T}};\ x^n = a^n\}$, $a^n \in A^n$, and where $G \in \mathcal{B}_B^n$. We have for the Feinstein codes that

$$\max_i \hat{\nu}^n(\Gamma_i^c | w_i) \le \epsilon.$$

We use the same codewords w_i for the channel code, but we now use the expanded regions $(\Gamma_i)_\delta$ for the decoding regions. Since the Feinstein codes were δ-robust, these sets are disjoint and the code well-defined. Since the channel is $\overline{d}$-continuous we can choose an n large enough to ensure that if $x^n = \overline{x}^n$, then

$$\overline{d}_n(\nu_x^n, \nu_{\overline{x}}^n) \le \delta^2.$$

Suppose that we have a Feinstein code such that for the induced channel

$$\hat{\nu}(\Gamma_i | w_i) \ge 1 - \epsilon.$$

Then if the conditions of Lemma 5.7 are met and μ_n is the channel input source of the Feinstein code, then

$$\begin{aligned} \hat{\nu}^n(\Gamma_i | w_i) &= \frac{1}{\mu_n^n(w_i)} \int_{c(w_i)} \nu_x^n(\Gamma_i)\, d\mu(x) \le \sup_{x \in c(w_i)} \nu_x^n(\Gamma_i) \\ &\le \inf_{x \in c(w_i)} \nu_x^n((\Gamma_i)_\delta) + \delta \end{aligned}$$

and hence

$$\inf_{x \in c(w_i)} \nu_x^n((\Gamma_i)_\delta) \ge \hat{\nu}^n(\Gamma_i | w_i) - \delta \ge 1 - \epsilon - \delta.$$

Thus if the channel block code is constructed using the expanded decoding sets, we have that

$$\max_i \sup_{x \in c(w_i)} \nu_x((\Gamma_i)_\delta^c) \le \epsilon + \delta;$$

that is, the code $\{w_i, (\Gamma_i)_\delta\}$ is a $(\lfloor e^{nR} \rfloor, n, \epsilon + \delta)$ channel code. We have now proved the following result.

Lemma 14.10. *Let ν be a stationary $\overline{d}$-continuous channel and C_n; $n \ge n_0$, a sequence of δ-robust $(\mu_n, \lfloor e^{nR} \rfloor, n, \epsilon)$ Feinstein codes. Then for n_1 sufficiently large and each $n \ge n_1$ there exists a $(\lfloor e^{nR} \rfloor, n, \epsilon + \delta)$ block channel code.*

Combining the lemma with Lemma 14.6 and Lemma 14.7 yields the following theorem.

Theorem 14.1. *Let ν be an AMS ergodic $\overline{d}$-continuous channel. If $R < C$ then given $\epsilon > 0$ there is an n_0 such that for all $n \geq n_0$ there exist $(\lfloor e^{nR} \rfloor, n, \epsilon)$ channel codes. If the channel is not ergodic, then the same holds true if C is replaced by C^*.*

Up to this point the channel coding theorems have been "one shot" theorems in that they consider only a single use of the channel. In a communication system, however, a channel will be used repeatedly in order to communicate a sequence of outputs from a source.

14.7 Joint Source and Channel Block Codes

We can now combine a source block code and a channel block code of comparable rates to obtain a block code for communicating a source over a noisy channel. Suppose that we wish to communicate a source $\{X_n\}$ with a distribution μ over a stationary and ergodic $\overline{d}$-continuous channel $[B, \nu, \hat{B}]$. The channel coding theorem states that if K is chosen to be sufficiently large, then we can reliably communicate length K messages from a collection of $\lfloor e^{KR} \rfloor$ messages if $R < C$. Suppose that $R = C - \epsilon/2$. If we wish to send the given source across this channel, then instead of having a source coding rate of $(K/N) \log \|B\|$ bits or nats per source symbol for a source (N, K) block code, we reduce the source coding rate to slightly less than the channel coding rate R, say $R_{\text{source}} = (K/N)(R - \epsilon/2) = (K/N)(C - \epsilon)$. We then construct a block source codebook C of this rate with performance near the operational DRF $\delta(R_{\text{source}}, \mu)$ defined in (12.1). Every codeword in the source codebook is assigned a channel codeword as index. The source is encoded by selecting the minimum distortion word in the codebook and then inserting the resulting channel codeword into the channel. The decoder then uses its decoding sets to decide which channel codeword was sent and then puts out the corresponding reproduction vector. Since the indices of the source code words are accurately decoded by the receiver with high probability, the reproduction vector should yield performance near that of $\delta((K/N)(C - \epsilon), \mu)$. Since ϵ is arbitrary and $\delta(R, \mu)$ is a continuous function of R, this implies that the optimal achievable performance for block coding μ for ν is given by $\delta((K/N)C, \mu)$, that is, by the operational distortion-rate function for block coding a source evaluated at the channel capacity normalized to bits or nats per source symbol. Making this argument precise yields the block joint source and channel coding theorem.

A *joint source and channel* (N, K) *block code* consists of an encoder $\alpha : A^N \to B^K$ and decoder $\beta : \hat{B}^K \to \hat{A}^N$. It is assumed that N source time units correspond to K channel time units. The block code yields

sequence coders $\overline{\alpha} : A^{\mathbb{T}} \to B^{\mathbb{T}}$ and $\overline{\beta} : \hat{B}^{\mathbb{T}} \to \hat{A}^{\mathbb{T}}$ defined by

$$\overline{\alpha}(x) = \{\alpha(x_{iN}^N);\ \text{all } i\}$$
$$\overline{\beta}(x) = \{\beta(x_{iN}^N);\ \text{all } i\}.$$

Let $\mathcal{E}$ denote the class of all such codes (all N and K consistent with the physical stationarity requirement). Let $\Delta^*(\mu, \nu, \mathcal{E})$ denote the block coding operational distortion-rate function function and $D(R, \mu)$ the Shannon distortion-rate function of the source with respect to an additive fidelity criterion $\{\rho_n\}$. We assume also that ρ_n is bounded, that is, there is a finite value $\rho_{\max}$ such that

$$\frac{1}{n}\rho_n(x^n, \hat{x}^n) \le \rho_{\max}$$

for all n. This assumption is an unfortunate restriction, but it yields a simple proof of the basic result.

Theorem 14.2. *Let $\{X_n\}$ be a stationary source with distribution μ and let ν be a stationary and ergodic $\overline{d}$-continuous channel with channel capacity C. Let $\{\rho_n\}$ be a bounded additive fidelity criterion. Given $\epsilon > 0$ there exists for sufficiently large N and K (where K channel time units correspond to N source time units) an encoder $\alpha : A^N \to B^K$ and decoder $\beta : \hat{B}^K \to \hat{A}^N$ such that if $\overline{\alpha} : A^{\mathbb{T}} \to B^{\mathbb{T}}$ and $\overline{\beta} : \hat{B}^{\mathbb{T}} \to \hat{A}^{\mathbb{T}}$ are the induced sequence coders, then the resulting performance is bounded above as*

$$\Delta(\mu, \overline{\alpha}, \nu, \overline{\beta}) = E\rho_N(X^N, \hat{X}^N) \le \delta(\frac{K}{N}C, \mu) + \epsilon.$$

Proof: Given ϵ, choose $\gamma > 0$ so that

$$\delta(\frac{K}{N}(C - \gamma), \mu) \le \delta(\frac{K}{N}C, \mu) + \frac{\epsilon}{3}$$

and choose N large enough to ensure the existence of a source codebook C of length N and rate $R_{\text{source}} = (K/N)(C - \gamma)$ with performance

$$\rho(C, \mu) \le \delta(R_{\text{source}}, \mu) + \frac{\epsilon}{3}.$$

We also assume that N (and hence also K) is chosen large enough so that for a suitably small δ (to be specified later) there exists a channel $(\lfloor e^{KR} \rfloor, K, \delta)$ code, with $R = C - \gamma/2$. Index the $\lfloor e^{NR_{\text{source}}} \rfloor$ words in the source codebook by the $\lfloor e^{K(C-\gamma/2} \rfloor$ channel codewords. By construction there are more indices than source codewords so that this is possible. We now evaluate the performance of this code.

Suppose that there are M words in the source codebook and hence M of the channel words are used. Let $\hat{x}_i$ and w_i denote corresponding

source and channel codewords, that is, if $\hat{x}_i$ is the minimum distortion word in the source codebook for an observed vector, then w_i is transmitted over the channel. Let Γ_i denote the corresponding decoding region. Then

$$\begin{aligned}
E\rho_N(X^N, \hat{X}^N) &= \sum_{i=1}^{M}\sum_{j=1}^{M}\int_{x:\alpha(x^N)=w_i} d\mu(x)\nu_x^K(\Gamma_j)\rho_N(x^N, \hat{x}_j) \\
&= \sum_{i=1}^{M}\int_{x:\alpha(x^N)=w_i} d\mu(x)\nu_x^K(\Gamma_i)\rho_N(x^N, \hat{x}_i) \\
&\quad + \sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M}\int_{x:\alpha(x^N)=w_i} d\mu(x)\nu_x^K(\Gamma_j)\rho_N(x^N, \hat{x}_j) \\
&\leq \sum_{i=1}^{M}\int_{x:\alpha(x^N)=w_i} d\mu(x)\rho_N(x^N, \hat{x}_i) \\
&\quad + \sum_{i=1}^{M}\sum_{j=1,J\neq i}^{M}\int_{x:\alpha(x^N)=w_i} d\mu(x)\nu_x^K(\Gamma_j)\rho_N(x^N, \hat{x}_j)
\end{aligned}$$

The first term is bounded above by $\delta(R_{\text{source}}, \mu) + \epsilon/3$ by construction. The second is bounded above by $\rho_{\max}$ times the channel error probability, which is less than δ by assumption. If δ is chosen so that $\rho_{\max}\delta$ is less than $\epsilon/2$, the theorem is proved. □

Theorem 14.3. *Let $\{X_n\}$ be a stationary source source with distribution μ and let ν be a stationary channel with channel capacity C. Let $\{\rho_n\}$ be a bounded additive fidelity criterion. For any block stationary communication system (μ, f, ν, g), the average performance satisfies*

$$\Delta(\mu, f, \nu, g) \leq \int_x d\overline{\mu}(x) D(C, \overline{\mu}_x),$$

where $\overline{\mu}$ is the stationary mean of μ and $\{\overline{\mu}_x\}$ is the ergodic decomposition of $\overline{\mu}$, C is the capacity of the channel, and $D(R, \mu)$ the Shannon distortion-rate function.

Proof: Suppose that the process $\{X_{nN}^N, U_{nK}^K, Y_{nK}^K, \hat{X}_{nN}^N\}$ is stationary and consider the overall mutual information rate $\overline{I}(X; \hat{X})$. From the data processing theorem (Lemma 8.7)

$$\overline{I}(X; \hat{X}) \leq \frac{K}{N}\overline{I}(U; Y) \leq \frac{K}{N}C.$$

Choose L sufficiently large so that

$$\frac{1}{n}I(X^n; \hat{X}^n) \leq \frac{K}{N}C + \epsilon$$

and

$$D_n(\frac{K}{N}C + \epsilon, \mu) \geq D(\frac{K}{N}C + \epsilon, \mu) - \delta$$

for $n \geq L$. Then if the ergodic component μ_x is in effect, the performance can be no better than

$$E_{\mu_x}\rho_N(X^n, \hat{X}^N) \geq \inf_{p^N \in \mathcal{R}_N(\frac{K}{N}C+\epsilon, \mu_x^N)} \rho_N(X^N, \hat{X}^N) \geq D_N(\frac{K}{N}C + \epsilon, \mu_x)$$

which when integrated yields a lower bound of

$$\int d\mu(x) D(\frac{K}{N}C + \epsilon, \mu_x) - \delta.$$

Since δ and ϵ are arbitrary, the lemma follows from the continuity of the distortion rate function. □

Combining the previous results yields the block coding optimal achievable performance for stationary sources and stationary and ergodic $\overline{d}$-continuous channels.

Corollary 14.6. *Let $\{X_n\}$ be a stationary source with distribution μ and let ν be a stationary and ergodic $\overline{d}$-continuous channel with channel capacity C. Let $\{\rho_n\}$ be a bounded additive fidelity criterion. The block coding operational DRF of (5.14) is given by*

$$\Delta(\mu, \nu, \mathcal{E}, \mathcal{D}) = \int d\overline{\mu}(x) D(C, \overline{\mu}_x).$$

14.8 Synchronizing Block Channel Codes

As in the source coding case, the first step towards proving a sliding block coding theorem is to show that a block code can be synchronized, that is, that the decoder can determine (at least with high probability) where the block code words begin and end. Unlike the source coding case, this cannot be accomplished by the use of a simple synchronization sequence which is prohibited from appearing within a block code word since channel errors can cause an unintended appearance of the sync word at the receiver. The basic idea still holds, however, if the codes are designed so that it is very unlikely that a non-sync word can be converted into a valid sync word. If the channel is $\overline{d}$-continuous, then good robust Feinstein codes as in Corollary 14.5 can be used to obtain good codebooks . The basic result of this section is Lemma 14.11 which states that given a sequence of good robust Feinstein codes, the code length can be chosen large enough to ensure that there is a sync word for a slightly modified codebook; that is, the synch word has length a speci-

fied fraction of the codeword length and the sync decoding words never appear as a segment of codeword decoding words. The technique is due to Dobrushin [33] and is an application of Shannon's random coding technique. The lemma originated in [68].

The basic idea of the lemma is this: In addition to a good long code, one selects a short good robust Feinstein code (from which the sync word will be chosen) and then performs the following experiment. A word from the short code and a word from the long code are selected independently and at random. The probability that the short decoding word appears in the long decoding word is shown to be small. Since this average is small, there must be at least one short word such that the probability of its decoding word appearing in the decoding word of a randomly selected long code word is small. This in turn implies that if all long decoding words containing the short decoding word are removed from the long code decoding sets, the decoding sets of most of the original long code words will not be changed by much. In fact, one must remove a bit more from the long word decoding sets in order to ensure the desired properties are preserved when passing from a Feinstein code to a channel codebook.

Lemma 14.11. *Assume that $\epsilon \le 1/4$ and $\{C_n; n \ge n_0\}$ is a sequence of ϵ-robust $\{\tau, M(n), n, \epsilon/2\}$ Feinstein codes for a $\overline{d}$-continuous channel ν having capacity $C > 0$. Assume also that $h_2(2\epsilon) + 2\epsilon \log(\|B\| - 1) < C$, where B is the channel output alphabet. Let $\delta \in (0, 1/4)$. Then there exists an n_1 such that for all $n \ge n_1$ the following statements are true.*

(A) *If $C_n = \{v_i, \Gamma_i;\ i = 1, \ldots, M(n)\}$, then there is a modified codebook $\mathcal{W}_n = \{w_i; W_i;\ i = 1, \ldots, K(n)\}$ and a set of $K(n)$ indices $\mathcal{K}_n = \{k_1, \cdots, k_{K(n)} \subset \{1, \cdots, M(n)\}$ such that $w_i = v_{k_i}$, $W_i \subset (\Gamma_i)_{\epsilon^2};\ i = 1, \ldots, K(n)$, and*

$$\max_{1 \le j \le K(n)} \sup_{x \in c(w_j)} \nu_x^n(W_j^c) \le \epsilon. \tag{14.24}$$

(B) *There is a sync word $\sigma \in A^r$, $r = r(n) = \lceil \delta n \rceil$ = smallest integer larger than δn, and a sync decoding set $S \in \mathcal{B}_B^r$ such that*

$$\sup_{x \in c(\sigma)} \nu_x^r(S^c) \le \epsilon. \tag{14.25}$$

and such that no r-tuple in S appears in any n-tuple in W_i; that is, if $G(b^r) = \{y^n : y_i^r = b^r \text{ some } i = 0, \ldots, n-r\}$ and $G(S) = \bigcup_{b^r \in S} G(b^r)$, then

$$G(S) \bigcap W_i = \varnothing, i = 1, \ldots, K(n). \tag{14.26}$$

(C) *We have that*

$$\|\{k : k \notin \mathcal{K}_n\}\| \le \epsilon \delta M(n). \tag{14.27}$$

The modified code $\mathcal{W}_n$ has fewer words than the original code C_n, but (14.27) ensures that $\mathcal{W}_n$ cannot be much smaller since

$$K(n) \geq (1 - \epsilon\delta)M(n). \tag{14.28}$$

Given a codebook $\mathcal{W}_n = \{w_i, W_i; i = 1, \ldots, K(n)\}$, a sync word $\sigma \in A^r$, and a sync decoding set S, we call the length $n+r$ codebook $\{\sigma \times w_i, S \times W_i; i = 1, \ldots, K(n)\}$ a *prefixed* or *punctuated* codebook.

Proof: Since ν is $\overline{d}$-continuous, n_2 can be chosen so large that for $n \geq n_2$

$$\max_{a^n \in A^n} \sup_{x,x' \in c(a^n)} \overline{d}_n(\nu_x^n, \nu_{x'}^n) \leq (\frac{\delta\epsilon}{2})^2. \tag{14.29}$$

From Corollary 14.5 there is an n_3 so large that for each $r \geq n_3$ there exists an $\epsilon/2$-robust $(\tau, J, r, \epsilon/2)$-Feinstein code $C_s = \{s_j, S_j : j = 1, \ldots, J\}$; $J \geq 2^{rR_s}$, where $R_s \in (0, C - h_2(2\epsilon) - 2\epsilon \log(\|B\| - 1))$. Assume that n_1 is large enough to ensure that $\delta n_1 \geq n_2$; $\delta n_1 \geq n_3$, and $n_1 \geq n_0$. Let 1_F denote the indicator function of the set F and define λ_n by

$$\begin{aligned}
&\lambda_n \\
&= J^{-1} \sum_{j=1}^{J} \frac{1}{M(n)} \sum_{i=1}^{M(n)} \hat{\nu}^n(G((S_j)_\epsilon) \bigcap \Gamma_i | v_i) \\
&= J^{-1} \sum_{j=1}^{J} \frac{1}{M(n)} \sum_{i=1}^{M(n)} \sum_{b' \in (S_j)_\epsilon} \sum_{y^n \in \Gamma_i} \hat{\nu}^n(y^n | v_i) 1_{G(b')}(y^n) \\
&= J^{-1} \frac{1}{M(n)} \sum_{i=1}^{M(n)} \sum_{y^n \in \Gamma_i} \hat{\nu}^n(y^n | v_i) \left[\sum_{j=1}^{J} \sum_{b' \in (S_j)_\epsilon} 1_{G(b')}(y^n) \right].
\end{aligned} \tag{14.30}$$

Since the $(S_j)_\epsilon$ are disjoint and a fixed y^n can belong to at most $n - r \leq n$ sets $G(b^r)$, the bracket term above is bound above by n and hence

$$\lambda_n \leq \frac{n}{J} \frac{1}{M(n)} \sum_{i=1}^{M(n)} \hat{\nu}^n(y^n | v_i) \leq \frac{n}{J} \leq n 2^{-rR_s} \leq n 2^{-\delta n R_s} \underset{n \to \infty}{\to} 0$$

so that choosing n_1 also so that $n_1 2^{-\delta n R_s} \leq (\delta\epsilon)^2$ we have that $\lambda_n \leq (\delta\epsilon)^2$ if $n \geq n_1$. From (14.30) this implies that for $n \geq n_1$ there must exist at least one j for which

$$\sum_{i=1}^{M(n)} \hat{\nu}^n(G((S_j)_\epsilon) \bigcap \Gamma_i | v_i) \leq (\delta\epsilon)^2$$

which in turn implies that for $n \geq n_1$ there must exist a set of indices $\mathcal{K}_n \subset \{1, \cdots, M(n)\}$ such that

$$\hat{\nu}^n(G((S_j)_\epsilon) \bigcap \Gamma_i | \nu_i) \le \delta\epsilon, i \in \mathcal{K}_n, \|\{i : i \notin \mathcal{K}_n\}\| \le \delta\epsilon.$$

Define $\sigma = s_j$; $S = (S_j)_{\epsilon/2}$, $w_i = v_{k_i}$, and $W_i = (\Gamma_{k_i} \bigcap G((S_j)_\epsilon)^c)_{\epsilon\delta}$; $i = 1, \ldots, K(n)$. We then have from Lemma 14.10 and (14.29) that if $x \in c(\sigma)$, then since $\epsilon\delta \le \epsilon/2$

$$\nu_x^r(S) = \nu_x^r((S_j)_{\epsilon/2}) \ge \hat{\nu}^r(S_j|\sigma) - \frac{\epsilon}{2} \ge 1 - \epsilon,$$

proving (14.25). Next observe that if $y^n \in (G((S_j)_\epsilon)^c)_{\epsilon\delta}$, then there is a $b^n \in G((S_j)_\epsilon)^c$ such that $d_n(y^n, b^n) \le \epsilon\delta$ and thus for $i = 0, 1, \ldots, n-r$ we have that

$$d_r(y_i^r, b_i^r) \le \frac{n}{r}\frac{\epsilon\delta}{2} \le \frac{\epsilon}{2}.$$

Since $b^n \in G((S_j)_\epsilon)^c$, it has no r-tuple within ϵ of an r-tuple in S_j and hence the r-tuples y_i^r are at least $\epsilon/2$ distant from S_j and hence $y^n \in H((S)_{\epsilon/2})^c)$. We have therefore that $(G((S_j)_\epsilon)^c)_{\epsilon\delta} \subset G((S_j)_\epsilon)^c$ and hence

$$\begin{aligned} G(S) \bigcap W_i &= G((S_j)_\epsilon) \bigcap (\Gamma_{k_i} \bigcap G((S_j)_\epsilon)^c)_{\delta\epsilon} \\ &\subset G((S_j)_{\epsilon/2}) \bigcap (G((S_j)_\epsilon)^c)_{\delta\epsilon} = \varnothing, \end{aligned}$$

completing the proof. □

Combining the preceding lemma with the existence of robust Feinstein codes at rates less than capacity (Lemma 14.10) we have proved the following synchronized block coding theorem.

Corollary 14.7. *Le ν be a stationary ergodic $\overline{d}$-continuous channel and fix $\epsilon > 0$ and $R \in (0, C)$. Then there exists for sufficiently large blocklength N, a length N codebook $\{\sigma \times w_i, S \times W_i;\ i = 1, \ldots, M\}$, $M \ge 2^{NR}$, $\sigma \in A^r$, $w_i \in A^n$, $r + n = N$, such that*

$$\begin{aligned} \sup_{x \in c(\sigma)} \nu_x^r(S^c) &\le \epsilon, \\ \max_{i \le j \le M} \nu_x^n(W_j^c) &\le \epsilon, \\ W_j \bigcap G(S) &= \varnothing. \end{aligned}$$

Proof: Choose $\delta \in (0, \epsilon/2)$ so small that $C - h(2\delta) - 2\delta \log(\|B\| - 1) > (1 + \delta)R(1 - \log(1 - \delta^2))$ and choose $R' \in ((1 + \delta)R(1 - \log(1 - \delta^2)), C - h(2\delta) - 2\delta \log(\|B\| - 1)$. From Lemma 14.10 there exists an n_0 such that for $n \ge n_0$ there exist δ-robust (τ, μ, n, δ) Feinstein codes with $M(n) \ge 2^{nR'}$. From Lemma 14.11 there exists a codebook $\{w_i, W_i;\ i = 1, \ldots, K(n)\}$, a sync word $\sigma \in A^r$, and a sync decoding set $S \in \mathcal{B}_B^r$, $r = \lceil \delta n \rceil$ such that

$$\max_j \sup_{x \in c(w_j)} \nu_x^n(W_j^c) \le 2\delta \le \epsilon,$$
$$\sup_{x \in c(\sigma)} \nu_x^r(S) \le 2\delta \le \epsilon,$$

$G(S) \bigcap W_j = \varnothing$; $j = 1, \ldots, K(n)$, and from (14.28) $M = K(n) \ge (1 - \delta^2)M(n)$. Therefore for $N = n + r$

$$\begin{aligned} N^{-1} \log M &\ge (n\lceil n\delta \rceil)^{-1} \log((1 - \delta^2)2^{nR'}) \\ &= \frac{nR' + \log(1 - \delta^2)}{n + n\delta} = \frac{R' + n^{-1}\log(1 - \delta^2)}{1 + \delta} \\ &\ge \frac{R' + \log(1 - \delta^2)}{1 + \delta} \ge R, \end{aligned}$$

completing the proof. □

14.9 Sliding-block Source and Channel Coding

Analogous to the conversion of block source codes into sliding-block source codes, the basic idea of constructing a sliding-block channel code is to use a punctuation sequence to stationarize a block code and to use sync words to locate the blocks in the decoded sequence. The sync word can be used to mark the beginning of a codeword and it will rarely be falsely detected during a codeword. Unfortunately, however, an r-tuple consisting of a segment of a sync and a segment of a codeword may be erroneously detected as a sync with nonnegligible probability. To resolve this confusion we look at the relative frequency of sync-detects over a sequence of blocks instead of simply trying to find a single sync. The idea is that if we look at enough blocks, the relative frequency of the sync-detects in each position should be nearly the probability of occurrence in that position and these quantities taken together give a pattern that can be used to determine the true sync location. For the ergodic theorem to apply, however, we require that blocks be ergodic and hence we first consider totally ergodic sources and channels and then generalize where possible.

Totally Ergodic Sources

Lemma 14.12. *Let ν be a totally ergodic stationary $\overline{d}$-continuous channel. Fix $\epsilon, \delta > 0$ and assume that $C_N = \{\sigma \times w_i; S \times W_i; i = 1, \ldots, K\}$ is a prefixed codebook satisfying (14.24)-(14.26). Let $\gamma_n : G^N \to C_N$ assign an N-tuple in the prefixed codebook to each N-tuple in G^N and let $[G, \mu, U]$*

be an N-stationary, N-ergodic source. Let $c(a^n)$ denote the cylinder set or rectangle of all sequences $u = (\cdots, u_{-1}, u_0, u_1, \cdots)$ for which $u^n = a^n$. There exists for sufficiently large L (which depends on the source) a sync locating function $s : B^{LN} \to \{0, 1, \ldots, N-1\}$ and a set $\Phi \in \mathcal{B}_G^m$, $m = (L+1)N$, such that if $u^m \in \Phi$ and $\gamma_N(U_{LN}^N) = \sigma \times w_i$, then

$$\inf_{x \in c(\gamma_m(u^m))} \nu_x(\gamma : s(\gamma^{LN}) = \theta, \theta = 0, \ldots, N-1; \gamma_{LN} \in S \times W_i) \geq 1 - 3\epsilon. \tag{14.31}$$

Comments: The lemma can be interpreted as follows. The source is block encoded using γ_N. The decoder observes a possible sync word and then looks "back" in time at previous channel outputs and calculates $s(\gamma^{LN})$ to obtain the exact sync location, which is correct with high probability. The sync locator function is constructed roughly as follows: Since μ and ν are N-stationary and N-ergodic, if $\overline{\gamma} : A^\infty \to B^\infty$ is the sequence encoder induced by the length N block code γ_N, then the encoded source $\mu\overline{\gamma}^{-1}$ and the induced channel output process η are all N-stationary and N-ergodic. The sequence $z_j = \eta(T^j c(S)))$; $j = \ldots, -1, 0, 1, \ldots$ is therefore periodic with period N. Furthermore, z_j can have no smaller period than N since from (14.24)–(14.26) $\eta(T^j c(S)) \leq \epsilon$, $j = r+1, \ldots, n-r$ and $\eta(c(S)) \geq 1 - \epsilon$. Thus defining the sync pattern $\{z_j;\ j = 0, 1, \ldots, N-1\}$, the pattern is distinct from any cyclic shift of itself of the form $\{z_k, \cdots, z_{N-1}, z_0, \cdots, x_{k-1}\}$, where $k \leq N-1$. The sync locator computes the relative frequencies of the occurrence of S at intervals of length N for each of N possible starting points to obtain, say, a vector $\hat{z}^N = (\hat{z}_0, \hat{z}_1, \cdots, \hat{z}_{N-1})$. The ergodic theorem implies that the $\hat{z}_i$ will be near their expectation and hence with high probability $(\hat{z}_0, \cdots, \hat{z}_{N-1}) = (z_\theta, z_{\theta+1}, \cdots, z_{N-1}, z_0, \cdots, z_{\theta-1})$, determining θ. Another way of looking at the result is to observe that the sources ηT^j; $j = 0, \ldots, N-1$ are each N-ergodic and N-stationary and hence any two are either identical or orthogonal in the sense that they place all of their measure on disjoint N-invariant sets. (See, e.g., Exercise 1, Section 6.7 of [55] or Section 8.2 of [58].) No two can be identical, however, since if $\eta T^i = \eta T^j$ for $i \neq j$; $0 \leq i, j \leq N-1$, then η would be periodic with period $|i-j|$ strictly less than N, yielding a contradiction. Since membership in any set can be determined with high probability by observing the sequence for a long enough time, the sync locator attempts to determine which of the N distinct sources ηT^j is being observed. In fact, synchronizing the output is exactly equivalent to forcing the N sources ηT^j; $j = 0, 1, \ldots, N-1$ to be distinct N-ergodic sources. After this is accomplished, the remainder of the proof is devoted to using the properties of $\overline{d}$-continuous channels to show that synchronization of the output source when driven by μ implies that with high probability the channel output can be synchronized for all fixed input sequences in a set of high μ probability.

The lemma is stronger (and more general) than the similar results of Nedoma [130] and Vajda [180], but the extra structure is required for application to sliding-block decoding.

Proof: Choose $\zeta > 0$ so that $\zeta < \epsilon/2$ and

$$\zeta < \frac{1}{8} \min_{i,j:z_i \neq z_j} |z_i - z_j|. \tag{14.32}$$

For $\alpha > 0$ and $\theta = 0, 1, \ldots, N-1$ define the sets $\psi(\theta, \alpha) \in \mathcal{B}_B^{LN}$ and $\tilde{\psi}(\theta, \alpha) \in \mathcal{B}_B^m$, $m = (L+1)N$ by

$$\psi(\theta, \alpha) = \{y^{LN} : |\frac{1}{L-1} \sum_{i=0}^{L-2} 1_S(y_{j+iN}^r) - z_{\theta+j}| \leq \alpha;\ j = 0, 1, \ldots, N-1\}$$

$$\tilde{\psi}(\theta, \alpha) = B^\theta \times \psi(\theta, \alpha) \times B^{N-\theta}.$$

From the ergodic theorem L can be chosen large enough so that

$$\eta(\bigcap_{\theta=0}^{N-1} T^{-\theta} c(\psi(\theta, \zeta))) = \eta^m(\bigcap_{\theta=0}^{N-1} \tilde{\psi}(\theta, \zeta)) \geq 1 - \zeta^2. \tag{14.33}$$

Assume also that L is large enough so that if $x_i = x_i'$, $i = 0, \ldots, m-1$ then

$$\overline{d}_m(\nu_x^m, \nu_{x'}^m) \leq (\frac{\zeta}{N})^2. \tag{14.34}$$

From (14.33)

$$\begin{aligned} \zeta^2 \geq \eta^m((\bigcap_{\theta=0}^{N-1} \tilde{\psi}(\theta, \zeta))^c) &= \sum_{a^m \in G^m} \int_{c(a^m)} d\mu(u) \nu_{\overline{y}(u)}^m((\bigcap_{\theta=0}^{N-1} \tilde{\psi}(\theta, \zeta)^c)) \\ &= \sum_{a^m \in G^m} \mu^m(a^m) \hat{\nu}((\bigcap_{\theta=0}^{N-1} \tilde{\psi}(\theta, \zeta))^c | y_m(a^m)) \end{aligned}$$

and hence there must be a set $\Phi \in \mathcal{B}_B^m$ such that

$$\hat{\nu}^m((\bigcap_{\theta=0}^{N-1} \tilde{\psi}(\theta, \zeta))^c | y_m(a^m)) \leq \zeta, a^m \in \Phi, \tag{14.35}$$

$$\mu^m(\Phi) \leq \zeta. \tag{14.36}$$

Define the sync locating function $s : B^{LN} \to \{0, 1, \cdots, N-1\}$ as follows: Define the set $\psi(\theta) = \{y^{LN} \in (\psi(\theta, \zeta))_{2\zeta/N}\}$ and then define

$$s(y^{LN}) = \begin{cases} \theta & y^{LN} \in \psi(\theta) \\ 1 & \text{otherwise} \end{cases}$$

We show that s is well-defined by showing that $\psi(\theta) \subset \psi(\theta, 4\zeta)$, which sets are disjoint for $\theta = 0, 1, \ldots, N-1$ from (14.32). If $y^{LN} \in \psi(\theta)$, there is a $b^{LN} \in \psi(\theta, \zeta)$ for which $d_{LN}(y^{LN}, b^{LN}) \le 2\zeta/N$ and hence for any $j \in \{0, 1, \cdots, N-1\}$ at most $LN(2\zeta/N) = 2\zeta L$ of the consecutive nonoverlapping N-tuples y^N_{j+iN}, $i = 0, 1, \ldots, L-2$, can differ from the corresponding b^N_{j+iN} and therefore

$$|\frac{1}{L-1}\sum_{i=0}^{L-2} 1_S(y^r_{j+iN}) - z_{\theta+j}| \le |\frac{1}{L-1}\sum_{i=0}^{L-2} 1_S(b^r_{j+iN}) - z_{\theta+j}| + 2\zeta \le 3\zeta$$

and hence $y^{LN} \in \psi(\theta, 4\zeta)$. If $\tilde{\psi}(\theta)$ is defined to be $B^\theta \times \psi(\theta) \times B^{N-\theta} \in \mathcal{B}^m_B$, then we also have that

$$(\bigcap_{\theta=0}^{N-1} \tilde{\psi}(\theta, \zeta))_{\zeta/N} \subset \bigcap_{\theta=0}^{N-1} \tilde{\psi}(\theta)$$

since if $y^n \in (\bigcap_{\theta=0}^{N-1} \tilde{\psi}(\theta, \zeta))_{\zeta/N}$, then there is a b^m such that $b^{LN}_\theta \in \psi(\theta, \zeta)$; $\theta = 0, 1, \ldots, N-1$ and $d_m(y^m, b^m) \le \zeta/N$ for $\theta = 0, 1, \ldots, N-1$. This implies from Lemma 14.10 and (14.34)–(14.36) that if $x \in y^m(a^m)$ and $a^m \in \Phi$, then

$$\begin{aligned} \nu^m_x(\bigcap_{\theta=0}^{N-1} \tilde{\psi}(\theta)) &\ge \nu^m_x((\bigcap_{\theta=0}^{N-1} \tilde{\psi}(\theta, \zeta))_{\zeta/N}) \\ &\ge \hat{\nu}(\bigcap_{\theta=0}^{N-1} \tilde{\psi}(\theta, \zeta) | y^m(a^m)) - \frac{\zeta}{N} \\ &\ge 1 - \zeta - \frac{\zeta}{N} \ge 1 - \epsilon. \end{aligned} \tag{14.37}$$

To complete the proof, we use (14.24)–(14.26) and (14.37) to obtain for $a^m \in \Phi$ and $\gamma_m(a_N L^N) = \sigma \times w_i$ that

$$\begin{aligned} &\nu_x(y : s(y^{LN}_\theta) = \theta, \theta = 0, 1, \ldots, N-1; y^N_{LN} \in S \times W_i) \\ &\qquad \ge \nu^m_x(\bigcap_{\theta=0}^{N-1} \psi(\theta)) - \nu^N_{T^{-NL}x}(S \times W^c_i) \ge 1 - \epsilon - 2\epsilon. \end{aligned}$$

□

Next the prefixed block code and the sync locator function are combined with a random punctuation sequence of Lemma 2.12 to construct a good sliding-block code for a totally ergodic source with entropy less than capacity.

Lemma 14.13. *Given a $\overline{d}$-continuous totally ergodic stationary channel ν with Shannon capacity C, a stationary totally ergodic source $[G, \mu, U]$*

with entropy rate $H(\mu) < C$, and $\delta > 0$, there exists for sufficiently large n, m a sliding-block encoder $f : G^n \to A$ and decoder $g : B^m \to G$ such that $P_e(\mu, \nu, f, g) \le \delta$.

Proof: Choose R, $\overline{H} < R < C$, and fix $\epsilon > 0$ so that $\epsilon \le \delta/5$ and $\epsilon \le (R - \overline{H})/2$. Choose N large enough so that the conditions and conclusions of Corollary 14.7 hold. Construct first a joint source and channel block encoder γ_N as follows: From the asymptotic equipartition property (Lemma 4.2 or Section 4.5) there is an n_0 large enough to ensure that for $N \ge n_0$ the set

$$\begin{aligned} G_N &= \{u^N : |N^{-1}h_N(u) - \overline{H}| \ge \epsilon\} \\ &= \{u^N : e^{-N(\overline{H}+\epsilon)} \le \mu(u^N) \le e^{-N(\overline{H}-\epsilon)}\} \end{aligned} \tag{14.38}$$

has probability

$$\mu_{U^N}(G_N) \ge 1 - \epsilon. \tag{14.39}$$

Observe that if $M' = \|G_N\|$, then

$$2^{N(\overline{H}-\epsilon)} \le M' \le 2^{N(\overline{H}+\epsilon)} \le 2^{N(R-\epsilon)}. \tag{14.40}$$

Index the members of G_N as β_i; $i = 1, \ldots, M'$. If $u_N = \beta_i$, set $\gamma_N(u_N) = \sigma \times w_i$. Otherwise set $\gamma_N(u_N) = \sigma \times w_{M'+1}$. Since for large N, $2^{N(R-\epsilon)} + 1 \le 2^{NR}$, γ_N is well-defined. γ_N can be viewed as a synchronized extension of the almost noiseless code of Section 3.5. Define also the block decoder $\psi_N(\mathcal{Y}^N) = \beta_i$ if $\mathcal{Y}^N \in S \times W_i$; $i = 1, \ldots, M'$. Otherwise set $\psi_N(\mathcal{Y}^N) = \beta^*$, an arbitrary reference vector. Choose L so large that the conditions and conclusions of Lemma 14.12 hold for C and γ_N. The sliding-block decoder $g_m : B^m \to G$, $m = (L+1)N$, yielding decoded process $\hat{U}_k = g_m(Y_{k-NL}^m)$ is defined as follows: If $s(\mathcal{Y}_{k-NL}, \cdots, \mathcal{Y}_k - 1) = \theta$, form $b^N = \psi_N(\mathcal{Y}_{k-\theta}, \cdots, \mathcal{Y}_{k-\theta-N})$ and set $\hat{U}_k(\mathcal{Y}) = g_m(\mathcal{Y}_{k-NL}, \cdots, \mathcal{Y}_{k+N}) = b_\theta$, the appropriate symbol of the appropriate block.

The sliding-block encoder f will send very long sequences of block words with random spacing to make the code stationary. Let K be a large number satisfying $K\epsilon \ge L + 1$ so that $m \le \epsilon KN$ and recall that $N \ge 3$ and $L \ge 1$. We then have that

$$\frac{1}{KN} \le \frac{1}{3K} \le \frac{\epsilon}{6}. \tag{14.41}$$

Use Corollary 2.1 to produce a (KN, ϵ) punctuation sequence Z_n using a finite length sliding-block code of the input sequence. The punctuation process is stationary and ergodic, has a ternary output and can produce only isolated 0's followed by KN 1's or individual 2's. The punctuation sequence is then used to convert the block encoder γ_N into a sliding-block coder: Suppose that the encoder views an input sequence $u = \cdots, u_{-1}, u_0, u_1, \cdots$ and is to produce a single encoded symbol x_0. If u_0

is a 2, then the encoder produces an arbitrary channel symbol, say a^*. If x_0 is not a 2, then the encoder inspects u_0, u_{-1}, u_{-2} and so on into the past until it locates the first 0. This must happen within KN input symbols by construction of the punctuation sequence. Given that the first 1 occurs at, say, $Z_l = 1$, the encoder then uses the block code γ_N to encode successive blocks of input N-tuples until the block including the symbol at time 0 is encoded. The sliding-block encoder than produces the corresponding channel symbol x_0. Thus if $Z_l = 1$, then for some $J < K x_0 = (\gamma_N(u_{l+JN}))_{l \bmod N}$ where the subscript denotes that the (l mod N)th coordinate of the block codeword is put out. The final sliding-block code has a finite length given by the maximum of the lengths of the code producing the punctuation sequence and the code imbedding the block code γ_N into the sliding-block code.

We now proceed to compute the probability of the error event $\{u, y : \hat{U}_0(y) \neq U_0(u)\} = E$. Let E_u denote the section $\{y : \hat{U}_0(y) \neq U_0(u)\}$, $\overline{f}$ be the sequence coder induced by f, and $F = \{u : Z_0(u) = 0\}$. Note that if $u \in T^{-1}F$, then $Tu \in F$ and hence $Z_0(Tu) = Z_1(u)$ since the coding is stationary. More generally, if $uT^{-i}F$, then $Z_i = 0$. By construction any 1 must be followed by KN 1's and hence the sets $T^{-i}F$ are disjoint for $i = 0, 1, \ldots, KN-1$ and hence we can write

$$
\begin{aligned}
P_e = \Pr(U_0 \neq \hat{U}_0) = \mu\nu(E) &= \int d\mu(u)\nu_{\overline{f}(u)}(E_u) \\
&\leq \sum_{i=0}^{LN-1} \int_{T^{-i}F} d\mu(u)\nu_{\overline{f}(u)}(E_u) + \sum_{i=LN}^{KN-1} \int_{T^{-i}F} d\mu(u)\nu_{\overline{f}(u)}(E_u) \\
&\qquad + \int_{(\bigcup_{i=0}^{KN-1} T^{-i}F)^c} d\mu(u) \\
&= LN\mu(F) + \sum_{i=LN}^{KN-1} \int_{T^{-i}F} d\mu(u)\nu_{\overline{f}(u)}(E_u) + \epsilon a \leq 2\epsilon \\
+ \sum_{i=LN}^{KN-1} \sum_{a^{kN} \in G^{kN}} &\int_{u' \in T^{-i}(F \cap c(a^K N))} d\mu(u')\nu_{\overline{f}(u')}(y' : U_0(u') \neq \hat{U}_0(u')),
\end{aligned}
\tag{14.42}
$$

where we have used the fact that $\mu(F) \leq (KN)^{-1}$ (from Corollary 2.1) and hence $LN\mu(F) \leq L/K \leq \epsilon$. Fix $i = kN + j$; $0 \leq j \leq N-1$ and define $u = T^{j+LN}u'$ and $y = T^{j+LN}y'$, and the integrals become

$$\int_{u' \in T^{-i}(F \bigcap c(a^{KN}))} d\mu(u') \nu_{\overline{f}(u')}(y' : U_0(u') \neq g_m(Y_{-NL}^m(y'))$$

$$= \int_{u \in T^{-(k-L)N}(F \bigcap c(a^{KN}))} d\mu(u') \times$$

$$\nu_{\overline{f}(T^{-(j+LN)}u)}(y : U_0(T^{j+LN}u) \neq g_m(Y_{-}NL^m(T^{j+NL}y)))$$

$$= \int_{u \in T^{-(k-L)N}(F \bigcap c(a^{KN}))} d\mu(u') \nu_{\overline{f}(T^{-(j+LN)}u)}(y : u_{j+LN} \neq g_m(y_j^m))$$

$$= \int_{u \in T^{-(k-L)N}(F \bigcap c(a^{KN}))} d\mu(u')$$

$$\times \nu_{\overline{f}(T^{-(j+LN)}u)}(y : u_{LN}^N = \psi_N(y_{LN}^N) \text{ or } s(y_j^{LN} \neq j)). \quad (14.43)$$

If $u_{LN}^N = \beta_j \in G_N$, then $u_{LN}^N = \psi_N(y_{LN}^N)$ if $y_{LN}^N \in S \times W_i$. If $u \in T^{-(k-L)N}c(a^{KN})$, then $u^m = a_{(k-L)N}^m$ and hence from Lemma 14.12 and stationarity we have for $i = kN + j$ that

$$\sum_{a^{KN} \in G^{KN}} \int_{T^{-i}(c(a^{KN}) \bigcap F)} d\mu(u) \nu_{\overline{f}(u)}(E_u)$$

$$\leq 3\epsilon \times \sum_{\substack{a^{KN} \in G^{KN} \\ a_{(k-L)N}^m \in \Phi \bigcap (G^{LN} \times G_N)}} \mu(T^{-(k-L)N}(c(a^{KN}) \bigcap F))$$

$$+ \sum_{\substack{a^{KN} \in G^{KN} \\ a_{(k-L)N}^m \notin \Phi \bigcap (G^{LN} \times G_N)}} \mu(T^{-(k-L)N}(c(a^{KN}) \bigcap F))$$

$$\leq 3\epsilon \times \sum_{a^{KN} \in G^{KN}} \mu(c(a^{KN}) \bigcap F))$$

$$+ \sum_{a_{(k-L)N}^m \in \Phi^c \bigcup (G^{LN} \times G_N)^c} \mu(c(a^{KN}) \bigcap F))$$

$$\leq 3\epsilon\mu(F) + \mu(c(\Phi^c) \bigcap F) + \mu(c(G_N) \bigcap F). \quad (14.44)$$

Choose the partition in Lemmas 2.11–2.12 to be that generated by the sets $c(\Phi^c)$ and $c(G^N)$ (the partition with all four possible intersections of these sets or their complements). Then the above expression is bounded above by

$$\frac{3\epsilon}{NK} + \frac{\epsilon}{NK} + \frac{\epsilon}{NK} \leq 5\frac{\epsilon}{NK}$$

and hence from (14.42) $P_e \leq 5\epsilon \leq \delta$, which completes the proof. □

The lemma immediately yields the following corollary.

Corollary 14.8. *If ν is a stationary $\overline{d}$-continuous totally ergodic channel with Shannon capacity C, then any totally ergodic source $[G, \mu, U]$ with $H(\mu) < C$ is admissible.*

Ergodic Sources

If a prefixed blocklength N block code of Corollary 14.8 is used to block encode a general ergodic source $[G, \mu, U]$, then successive N-tuples from μ may not be ergodic, and hence the previous analysis does not apply. From the Nedoma ergodic decomposition [129] (see, e.g., [55], p. 232, or [58], p. 253), any ergodic source μ can be represented as a mixture of N-ergodic sources, all of which are shifted versions of each other. Given an ergodic measure μ and an integer N, then there exists a decomposition of μ into M N-ergodic, N-stationary components where M divides N, that is, there is a set $\Pi \in \mathcal{B}_G^\infty$ such that

$$\begin{aligned} T^M\Pi &= \Pi \\ \mu(T^i\Pi \bigcap T^j\Pi) &= 0;\ i, j \le M, i \ne j \\ \mu(\bigcup_{i=0}^{M-1} T^i\Pi) &= 1 \\ \mu(\Pi) &= \frac{1}{M}, \end{aligned}$$

such that the sources $[G, \mu_i, U]$, where

$$\pi_i(W) = \mu(W|T^i\Pi) = M\mu(W \bigcap T^i\Pi)$$

are N-ergodic and N-stationary and

$$\mu(W) = \frac{1}{M}\sum_{i=0}^{M-1} \pi_i(W) = \frac{1}{M}\sum_{i=0}^{M-1} \mu(W \bigcap T^i\Pi). \tag{14.45}$$

This decomposition provides a method of generalizing the results for totally ergodic sources to ergodic sources. Since $\mu(\cdot|\Pi)$ is N-ergodic, Lemma 14.13 is valid if μ is replaced by $\mu(\cdot|\Pi)$. If an infinite length sliding-block encoder f is used, it can determine the ergodic component in effect by testing for $T^{-i}\Pi$ in the base of the tower and insert i dummy symbols and then encode using the length N prefixed block code. In other words, the encoder can line up the block code with a prespecified one of the N-possible N-ergodic modes. A finite-length encoder can then be obtained by approximating the infinite-length encoder by a finite length encoder. Making these ideas precise yields the following result.

Theorem 14.4. *If ν is a stationary $\overline{d}$-continuous totally ergodic channel with Shannon capacity C, then any ergodic source $[G, \mu, U]$ with $H(\mu) < C$ is admissible.*

Proof: Assume that N is large enough for Corollary 14.7 and (14.38)–(14.40) to hold. From the Nedoma decomposition

$$\frac{1}{M}\sum_{i=0}^{M-1}\mu^N(G_N|T^i\Pi) = \mu^N(G_N) \ge 1-\epsilon$$

and hence there exists at least one i for which $\mu^N(G_N|T^i\Pi) \ge 1-\epsilon$; that is, at least one N-ergodic mode must put high probability on the set G_N of typical N-tuples for μ. For convenience relabel the indices so that this good mode is $\mu(\cdot|\Pi)$ and call it the design mode. Since $\mu(\cdot|\Pi)$ is N-ergodic and N-stationary, Lemma 14.12 holds with μ replaced by $\mu(\cdot|\Pi)$; that is, there is a source/channel block code (γ_N, ψ_N) and a sync locating function $s : B^{LN} \to \{0,1,\cdots,M-1\}$ such that there is a set $\Phi \in G_m$; $m = (L+1)N$, for which (14.31) holds and $\mu^m(\Phi|\Pi) \ge 1-\epsilon$. The sliding-block decoder is exactly as in Lemma 14.12. The sliding-block encoder, however, is somewhat different. Consider a punctuation sequence or tower as in Lemma 2.12, but now consider the partition generated by Φ, G_N, and $T^i\Pi$, $i = 0,1,\ldots,M-1$. The infinite length sliding-block code is defined as follows: If $u \notin \bigcup_{k=0}^{NK-1} T^k F$, then $f(u) = a^*$, an arbitrary channel symbol. If $u \in T^i(F \bigcap T^{-j}\Pi)$ and if $i < j$, set $f(u) = a^*$ (these are spacing symbols to force alignment with the proper N-ergodic mode). If $j \le i \le KN-(M-j)$, then $i = j+kN+r$ for some $0 \le k \le (K-1)N$, $r \le N-1$. Form $G_N(u_{j+kN}^N) = a^N$ and set $f(u) = a_r$. This is the same encoder as before, except that if $u \in T^j\Pi$, then block encoding is postponed for j symbols (at which time $u \in \Pi$). Lastly, if $KN-(M-j) \le i \le KN-1$, then $f(u) = a^*$.

As in the proof of Lemma 14.13

$$\begin{aligned}
P_e(\mu,\nu,f,g_m) &= \int d\mu(u)\nu_{f(u)}(y : U_0(u) \ne g_m(Y_{-LN}^m(y))) \\
&\le 2\epsilon + \sum_{i=LN}^{KN-1}\int u \in T^iF\, d\mu(u)\nu_{f(u)}(y : U_0(u) \ne \hat{U}_0(y)) \\
&= 2\epsilon + \sum_{i=LN}^{KN-1}\sum_{j=0}^{M-1}\sum_{a^{KN}\in G^{KN}} \\
&\int_{u\in T^i(c(a^{KN}))\bigcap F\bigcap T^{-j}\Pi)} d\mu(u)\nu_{f(u)}(y : U_0(u) \ne \hat{U}_0(y)) \\
&\le 2\epsilon + \sum_{j=0}^{M-1}\sum_{i=LN+j}^{KN-(M-j)}\sum_{a^{KN}\in G^{KN}} \\
&\int_{u\in T^i(c(a^{KN}))\bigcap F\bigcap T^{-j}\Pi)} d\mu(u)\nu_{f(u)}(y : U_0(u) \ne \hat{U}_0(y)) \\
&\qquad + \sum_{j=0}^{M-1} M\mu(F\bigcap T^{-j}\Pi), \quad (14.46)
\end{aligned}$$

where the rightmost term is

$$M \sum_{j=0}^{M-1} \mu(F \bigcap T^{-j}\Pi) \le \frac{M}{KN} \le \frac{1}{K} \le \epsilon.$$

Thus

$$P_e(\mu, \nu, f, g_m) \le 3\epsilon + \sum_{j=0}^{M-1} \sum_{i=LN+j}^{KN-(M-j)} \sum_{a^{KN} \in G^{KN}} \int_{u \in T^i(c(a^{KN}) \bigcap F \bigcap T^{-j}\Pi)} d\mu(u) \nu_{f(u)}(y : U_0(u) \ne \hat{U}_0(y)).$$

Analogous to (14.43) (except that here $i = j + kN + r$, $u = T^{-(LN+r)}u'$)

$$\int_{u' \in T^i(c(a^{KN}) \bigcap F \bigcap T^{-j}\Pi)} d\mu(u') \nu_{f(u')}(y' : U_0(u') = g_m(Y_{-LN}^m(y'))) \le \int_{T^{j+(k-L)N}(c(a^{KN}) \bigcap F \bigcap T^{-j}\Pi)} d\mu(u) \times \nu_{f(T^i + LNu)}(y : u_{LN}^N \ne \psi_N(y_{LN}^N) \text{ or } s(y_r^{LN}) \ne r).$$

Since $u \in T^{j+(k-L)N}(c(a^{KN}) \bigcap F \bigcap T^{-j}\Pi$ implies $u^m = a_{j+(k-L)N}^m$, analogous to (14.44) we have that for $i = j + kN + r$

$$\begin{aligned}
&\sum_{a^{KN} \in G^{KN}} \int_{T^i(c(a^{KN}) \bigcap F \bigcap T^{-j}\Pi)} d\mu(u) \nu_{f(u)}(y : U_0(u) \ne g_m(Y_{-LN}^m(y))) \\
&\quad = \epsilon \sum_{a^{KN} : a_{j+(k-L)N}^m \in \Phi} \mu(T^{j+(k-L)N}(c(a^{KN}) \bigcap F \bigcap T^{-j}\Pi)) \\
&\quad + \sum_{a^{KN} : a_{j+(k-L)N}^m \notin \Phi} \mu(T^{j+(k-L)N}(c(a^{KN}) \bigcap F \bigcap T^{-j}\Pi)) \\
&\quad = \epsilon \sum_{a^{KN} : a_{j+(k-L)N}^m \in \Phi} \mu(c(a^{KN}) \bigcap F \bigcap T^{-j}\Pi) \\
&\quad + \sum_{a^{KN} : a_{j+(k-L)N}^m \notin \Phi} \mu(c(a^{KN}) \bigcap F \bigcap T^{-j}\Pi) \\
&\quad = \epsilon \mu(T^{-(j+(k-L)N)} c(\Phi) \bigcap F \bigcap T^{-j}\Pi) \\
&\qquad + \mu(T^{-(j+(k-L)N)} c(\Phi)^c \bigcap F \bigcap T^{-j}\Pi).
\end{aligned}$$

From Lemma 2.12 (the Rohlin-Kakutani theorem), this is bounded above by

$$\epsilon \frac{\mu(T^{-(j+(k-L)N)}c(\Phi) \bigcap T^{-j}\Pi)}{KN} + \frac{\mu(T^{-(j+(k-L)N)}c(\Phi)^c \bigcap T^{-j}\Pi)}{KN}$$
$$= \epsilon \frac{\mu(T^{-(j+(k-L)N)}c(\Phi)|T^{-j}\Pi)\mu(\Pi)}{KN} + \frac{\mu(T^{-(j+(k-L)N)}c(\Phi)^c|T^{-j}\Pi)\mu(\Pi)}{KN}$$
$$= \epsilon\mu(c(\Phi)|\Pi)\frac{\mu(\Pi)}{KN}\mu(c(\Phi)^c|\Pi)\frac{\mu(\Pi)}{KN} + \leq \frac{2\epsilon}{MKN}.$$

With (14.45)–(14.46) this yields

$$P_e(\mu, \nu, f, g_m) \leq 3\epsilon + \frac{MKN2\epsilon}{MKN} \leq 5\epsilon, \tag{14.47}$$

which completes the result for an infinite sliding-block code.

The proof is completed by applying Corollary 5.2, which shows that by choosing a finite length sliding-block code f_0 from Lemma 5.2 so that $\Pr(f \neq f_0)$ is sufficiently small, then the resulting P_e is close to that for the infinite length sliding-block code. □

The theorem can be combined with the sliding block source coding theorem to prove a joint source and channel coding theorem similar to Theorem 14.2, that is, one can show that given a source with distortion rate function $D(R)$ and a channel with capacity C, then sliding-block codes exist with average distortion approximately $D(C)$.

We have considered only discrete channels, which is less general than the continuous additive Gaussian noise channels considered in many classic information theory texts. On the other hand, we have considered more general memory structures than are usually encountered, and we have followed the common thread of the book to develop coding theorems for sliding-block codes as well as block codes.

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